

Why Have Interest Rates Fallen far Below the Return on Capital

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ABSTRACT

Risk-free rates have been falling since the 1980s. The return on capital, defined here as the profits over the stock of capital, has not. We analyze these trends in a calibrated OLG model designed to encompass many of the "usual suspects" cited in the debate on secular stagnation. Declining labor force and productivity growth imply a limited decline in real interest rates and deleveraging cannot account for the joint decline in the risk free rate and increase in the risk premium. If we allow for a change in the (perceived) risk to productivity growth to fit the data, we find that the decline in the risk-free rate requires an increase in the borrowing capacity of the indebted agents in the model, consistent with the increase in the sum of public and private debt since the crisis but at odds with a deleveraging-based explanation put forth in Eggertsson and Krugman (2012).⁴

Keywords: secular stagnation, interest rates, risk, return on capital.

JEL classification: E00, E40.

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The question we address is based on two stylized facts.

First, real interest rates have declined steadily over the last 2 decades. This downward trend is observed across OECD countries, for short-term and long-term interest rates as well as estimates of the natural rate of interest, and whatever the approach taken to approximate inflation expectations to define *ex ante* real interest rates. Following Summers (2014) many economists have associated this decline to secular stagnation. The lower growth rates of per capita GDP and the demographic transition imply lower investment and higher savings so much so that the return to savings should decline.

Second, the return to capital as measured from national accounts has remained flat. Gomme et al. (2011) build the return to productive capital as the net operating surplus, which is equal to value added minus depreciation and payments to labor, divided by the capital stock. Gomme et al. (2015) and Caballero et al. (2017) stress that, in the US, return to productive capital has no trend. It fluctuates with the cycle around 10 to 11 percent before tax and around 7 percent after tax. We compute similar indicators of the return on productive capital for the Euro area and the US from the AMECO database. Again, we see no downward trend in this measure of the return on investment. Altogether, we observe both a downward trend in real interest rates and stable return to productive capital (see Figure).

The possible explanations of a permanent decline in real interest rates have been largely discussed (Fischer 2016a, 2016b, Teulings and Baldwin 2014, Bean et al. 2015, Gourinchas et al. 2016) and include aging pressure on savings, income inequality, a slower pace of productivity, deleveraging, a collapse in the relative price of investment, a shortage of safe assets and an increase in the perception of risks. But so far there has been little quantitative evaluation of the competing explanations (but see Rachel and Smith 2015) and little of it model-based.

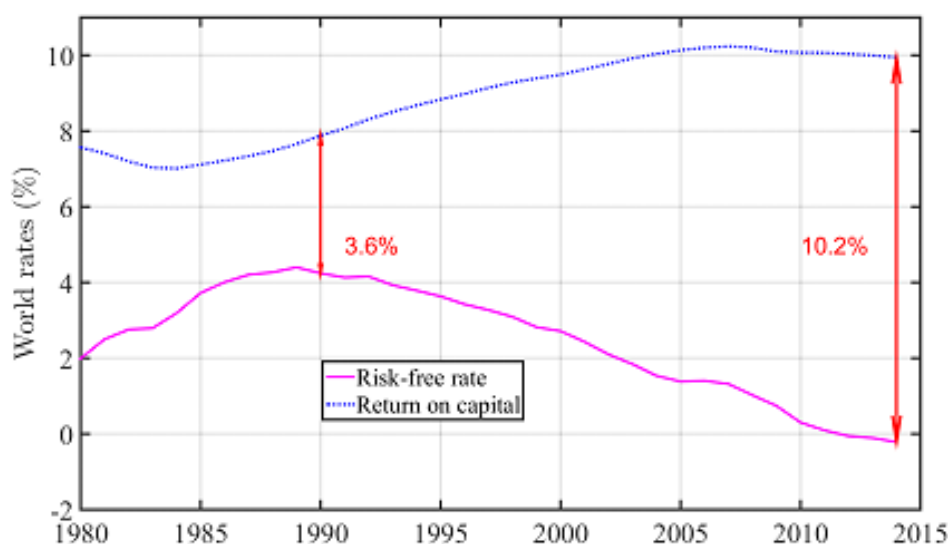
That's why we develop a framework combining Coeurdacier et al. (2015) and Eggertsson and Mehrotra (2014) to encompass most of the previous explanations for low interest rates. We extend the model to include risk, this allows to explain the divergence between risk-free rates and the return on capital; it also makes contact with the literature on safe assets (Caballero et al. 2008, Caballero and Farhi 2014) because safe assets only make sense in a context with risk.

The model is an overlapping generation where agents live three periods, and where there are two assets, the debt and a risky asset. Young agents borrow (safe asset) and face to a borrowing constraint, while middle-aged lend and buy capital (risky asset). There is uncertainty on productivity. The interest rate is then determined by the Euler equation of the savers, within which the constraints faced by the borrowers and other determinants enter through market clearing. Risk induces also a portfolio choice for the savers.

We calibrate this model for the US, the EA and a representation of the world (based on the aggregation of EA, US, China and Japan), we show that

- (i) Demographic and productivity trends cannot quantitatively explain the decrease of risk-free rates, jointly with stagnation of capital return.
- (ii) A quantitative explanation of the evolutions of these rates is an increase in uncertainty on productivity.
- (iii) This explanation does not require a decrease of the debt.

Increase in the risk-premium from 1990 to 2015



Note: risk-free rate is based on an average of real rates for the US and France extracted from Hamilton et al. (2016), return on capital is an average of the real return on capital for the US and the EA, computed thanks to the AMECO national accounts database (authors' computations of "profits/stock of capital")

Les taux d'intérêt ont baissé, pas le rendement du capital.

RÉSUMÉ

Les taux sans risque ont diminué depuis 1980, pas le rendement du capital, défini comme le ratio des profits rapportés au stock de capital. Nous analysons ces évolutions dans un modèle OLG calibré, prenant en compte beaucoup des facteurs traditionnellement avancés dans le débat de la stagnation séculaire. La baisse de la population active et de la productivité explique une faible partie de la baisse des taux d'intérêt, et le deleveraging ne peut pas expliquer conjointement la baisse du taux sans risque et l'augmentation de la prime de risque. Nous montrons qu'une augmentation du risque sur la croissance de la productivité permet de reproduire les données, sans nécessiter de baisse de la contrainte d'endettement. Cette explication va à l'encontre de l'idée de deleveraging mise en avant par Eggertsson et Krugman (2012) mais est cohérente avec l'augmentation des dettes publiques et privées depuis la Crise.

Mots-clés : stagnation séculaire, taux d'intérêt, risque, rendement du capital.

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1 Introduction

It has been nearly ten years since the “Global Financial Crisis” began.¹ Within two years of its inception, short-term nominal interest rates were driven to near-zero levels in the large advanced economies (U.S., Euro-area, UK, Japan) and have stayed there since. The Fed just recently “lifted off” while the UK hasn’t begun to consider when it would do so, and Japan and the Euro area have gone below zero. With low and (relatively) steady inflation, real rates have been negative for a while, and not just short-term rates but also rates at the 5-year and 10-year horizon (Hamilton et al. 2016).

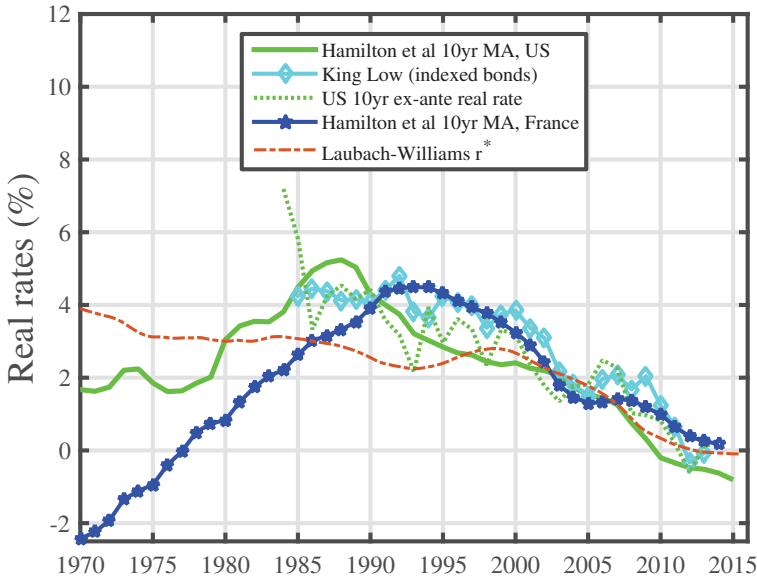


Figure 1: Real return on government bonds.

Much of the macroeconomic research responding to the financial crisis has taken place within the paradigm of the DSGE. Understanding the reasons for reaching the lower bound (the reason for low interest rate) was less urgent than understanding the proper responses to the situation. Also, the methodology relies on some approximation around a steady state, whether linear or nonlinear (Fernandez-Villaverde et al. 2012, Gust et al. 2012). Hence the low (real) interest rates are modeled as the result of an exogenous shock, for example to the discount rate or to a borrowing constraint (Eggertsson and Woodford 2003, Eggertsson and Krugman 2012), inducing deviations from a steady state whose dynamics (modified as needed by policy) are the core prediction of the model.

After a decade of low interest rates, the shock paradigm becomes less attractive because of the strains it places on the assumption of independent Gaussian shocks (Aruoba et al. 2013). Instead, a growing concern has emerged with a hypothesis dubbed by Summers (2014) “secular stagnation”: low interest rates may not be temporary deviations but

¹The beginning of the crisis is commonly dated to the closure of two Paribas funds in August 2007.

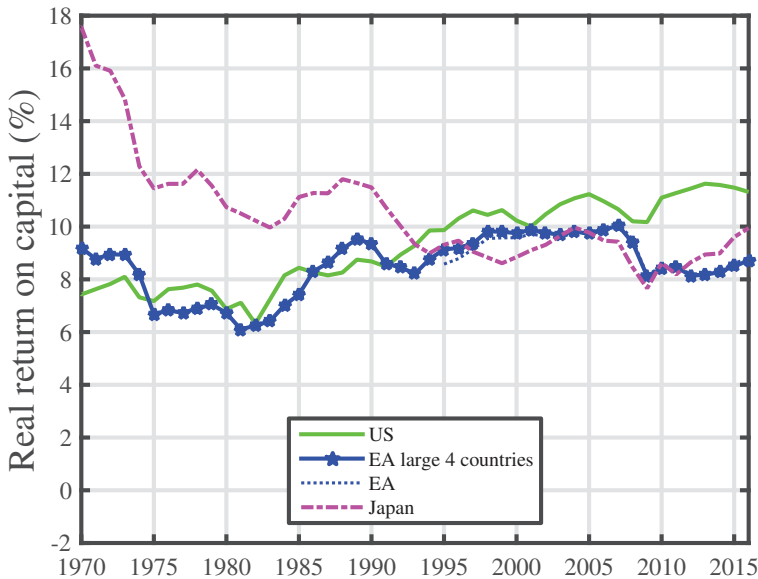


Figure 2: Return on capital.

a now-permanent state of affairs. Are low rates the “new normal” and can anything be done about it?

These are important policy questions (Fischer 2016a,b) and can only be addressed in a model. Three recent policy-oriented publications (Teulings and Baldwin 2014, Bean et al. 2015, Gourinchas et al. 2016) have recently collected the possible explanations for such a permanent decline in interest rates. These include aging pressure on savings, income inequality, a slower pace of productivity, deleveraging, a collapse in the relative price of investment, a shortage of safe assets and an increase in the perception or risks. But so far there has been little quantitative evaluation of the competing explanations (but see Rachel and Smith 2015) and little of it model-based.² Our question is simple: can we account for current low interest rates in a model that encompasses the most likely factors?

To answer it we develop a framework that combines Coeurdacier et al. (2015) and Eggertsson and Mehrotra (2014) to encompass most of the current explanations for low interest rates. Importantly, we extend the model to include risk. There are two reasons for this. One is to make contact with the literature on the shortage of safe assets (Caballero et al. 2008, Caballero and Farhi 2014) because safe assets only make sense in a context with risk. Second, we want to address an important fact on which we elaborate in the next section, namely the divergence between rates on (government) bonds, which have fallen, and the return on capital, which has not.

Using a single framework that encompasses the broad range of proposed explanations is like placing all the “usual suspects” in the same lineup. It comes at a cost if we want the

²Demographic factors have been evaluated in quantitative models (Carvalho et al. 2016, Gagnon et al. 2016). See also (Favero et al. 2016).

model to remain tractable: there are only three generations and only one source of risk.

2 Stylized facts

First, real interest rates have declined steadily over the last 2 decades (Figure 1). This downward trend is observed across OECD countries, for short-term and long-term interest rates as well as estimates of the natural rate of interest, and whatever the approach taken to approximate inflation expectations to define ex ante real interest rates (King and Low 2014, Hamilton et al. 2016, Rachel and Smith 2015, Laubach and Williams 2016, Holston et al. 2016, Fries et al. 2016, Fischer 2016a,b). In the case of the euro area, real rates were also negative in the 1970s. However this corresponds to a period of financial repression with limited openness of euro area financial markets. Hence we focus this paper on understanding the decline in real rates since 1990 rather than explaining their negative level in the euro area in the 1970s. Following Summers (2014) many economists have associated this decline to secular stagnation. The lower growth rates of per capita GDP and the demographic transition imply lower investment and higher savings so much so that the return to savings should decline.

Second, the return to capital as measured from national accounts has remained flat. Gomme et al. (2011) build the return to productive capital as the net operating surplus, which is equal to value added minus depreciation and payments to labor, divided by the capital stock. Gomme et al. (2015) and Caballero et al. (2017) stress that, in the US, return to productive capital has no trend. It fluctuates with the cycle around 10 to 11 percent before tax and around 7 percent after tax. In Figure 2 we report similar indicators of the return on productive capital for the Euro area and the US from the AMECO database. Again, we see no downward trend in this measure of the return on investment.³ Altogether, we observe both a downward trend in real interest rates and stable return to productive capital.

3 The Model

Many of the factors cited in Bean et al. (2015), Rachel and Smith (2015) and Gourinchas et al. (2016) can be embedded in the OLG model we present here, which nests Eggertsson and Mehrotra (2014) and Coeurdacier et al. (2015), and adds risk. The determination of the interest in those models comes down to the Euler equation of savers, within which

³The comparability of the data across countries is somewhat limited. In some countries, such as Italy and Germany, the income of unincorporated businesses which includes labor income of self employed are included (Garnier et al. 2015). In addition, and unlike the measure developed by Gomme et al. (2011, 2015), AMECO stock of capital includes dwellings and the flow of income to capital does not include rents. Caballero et al. (2017) who adjust their estimates for intangible intellectual property product introduced by the BEA since 2013 again find no evidence of a downward trend. However, there is little reason why either of these characteristics would modify the trend of the return to productive capital.

the constraints faced by borrower agents and other determinants enter through market-clearing. In the presence of risk, the savers also face a portfolio choice.

3.1 Description

In each discrete time period t a generation is born that lives 3 periods y, m, o . The size of the generation born at t is $N_t = g_{L,t} N_{t-1}$. Preferences are of the Epstein and Zin (1989) - Weil (1990) form:

$$V_t^y = \left(c_t^{y1-\rho} + \beta (E_t V_{t+1}^m)^{\frac{1-\rho}{1-\gamma}} \right)^{\frac{1-\gamma}{1-\rho}} \quad (1)$$

$$V_{t+1}^m = \left(c_{t+1}^{m1-\rho} + \beta (E_{t+1} V_{t+2}^o)^{\frac{1-\rho}{1-\gamma}} \right)^{\frac{1-\gamma}{1-\rho}} \quad (2)$$

$$V_{t+2}^o = c_{t+2}^{o1-\gamma} \quad (3)$$

with β the discount factor, $\gamma \geq 0$ the coefficient of relative risk aversion, $1/\rho \geq 1$ the intertemporal elasticity of substitution. The only source of risk is an aggregate productivity shock.

The factors of production are capital, which depreciates at a rate δ , and labor, supplied inelastically by young and middle-aged agents. The labor productivity of a member of generation t is e_t^y when young and $e_t^m = 1$ when middle-aged. The aggregate productivity of labor over time is $A_t = g_{A,t} A_{t-1}$ and is stochastic. The neo-classical constant-returns production function combines capital (with share α) and labor (with share $1 - \alpha$) to produce output, one unit of which can become either one unit of consumption or $1/p_t^k$ units of investment; the relative price of investment goods is exogenous and follows $p_t^k = g_{I,t} p_{t-1}^k$. Markets are competitive and prices are flexible. Labor earns a wage w_t while capital earns a return r_t^k .

Agents can purchase investment goods, and can also borrow from and lend to each other at a gross rate R_{t+1} , but they cannot owe more (principal and interest) than a fraction θ_t of next period's labor income. We will focus on situations in which the young borrow from middle-aged, and the middle-aged lend to the young and invest in physical capital by buying the depreciated stock in the hands of the old and purchasing investment goods.

The following equations summarize the above. Agents of generation t choose $\{c_t^y, c_{t+1}^m, c_{t+2}^o, k_{t+2}^m, b_{t+1}^y, b_{t+2}^m\}$ to maximize (1–3) subject to three budget constraints and one borrowing constraint:

$$c_t^y = b_{t+1}^y + w_t e_t^y \quad (4)$$

$$c_{t+1}^m - b_{t+2}^m + p_{t+1}^k k_{t+2}^m = w_{t+1} - R_{t+1} b_{t+1}^y \quad (5)$$

$$c_{t+2}^o = (p_{t+2}^k (1 - \delta) + r_{t+2}^k) k_{t+2}^m - R_{t+2} b_{t+2}^m \quad (6)$$

$$b_{t+1}^y \leq \theta_t E_t (w_{t+1} / R_{t+1}). \quad (7)$$

On the production side, the production function combines the current capital stock $N_{t-2}k_t^m$ (held by the old of generation $t - 2$ but chosen at $t - 1$ when they were middle-aged) and the labor supply of current young and middle-aged $e_t^y N_t + N_{t-1}$ to produce

$$Y_t = (N_{t-2}k_t^m)^\alpha [A_t(e_t^y N_t + N_{t-1})]^{1-\alpha}$$

which yields the wage rate and capital rental rate

$$w_t = (1 - \alpha)A_t^{1-\alpha}k_t^\alpha \quad (8)$$

$$r_t^k = \alpha A_t^{1-\alpha}k_t^{\alpha-1} \quad (9)$$

both written in terms of the capital/labor ratio k_t defined as

$$k_t \equiv \frac{N_{t-2}k_t^m}{e_t^y N_t + N_{t-1}} = \frac{k_t^m}{g_{L,t-1}(1 + e_t^y g_{L,t})}. \quad (10)$$

The final condition imposes clearing of the bond market at time t :

$$g_{L,t}b_{t+1}^y + b_{t+1}^m = 0. \quad (11)$$

3.2 Equilibrium conditions

The solution proceeds as follows. Following Giovannini and Weil (1989) we first express the middle-aged agent's first-order conditions in terms of a total return on their portfolio, and derive the demand for the two available assets, capital and loans to the young, as well as relation between the two returns. We then use market clearing: the demand for capital must equal the aggregate stock of capital, while the young's borrowing constraint, expressed in terms of their wages, determines the supply of the other asset. This allows us to derive the law of motion for the capital stock. We assume here that $\delta = 1$; the general case is treated in the appendix.

The problem of the middle-aged of generation $t - 1$ is

$$\max_{c_t^m, c_{t+1}^o} \left((c_t^m)^{1-\rho} + \beta E_t [c_{t+1}^o]^{1-\gamma} \right)^{\frac{1-\gamma}{1-\rho}}$$

subject to

$$c_t^m + p_t^k k_{t+1}^m - b_{t+1}^m = w_t - R_t b_t^y \quad (12)$$

$$c_{t+1}^o = R_{t+1} p_{t+1}^k k_{t+1}^m - R_{t+1} b_{t+1}^m \quad (13)$$

which leads to the first-order conditions

$$(c_t^m)^{-\rho} = \beta [E_t(c_{t+1}^o)^{1-\gamma}]^{\frac{\gamma-\rho}{1-\gamma}} E_t [(c_{t+1}^o)^{-\gamma} R_{t+1}^k] \quad (14)$$

$$(c_t^m)^{-\rho} = \beta [E_t(c_{t+1}^o)^{1-\gamma}]^{\frac{\gamma-\rho}{1-\gamma}} E_t [(c_{t+1}^o)^{-\gamma}] R_{t+1}. \quad (15)$$

To see it as a portfolio problem, express the budget constraints (12)–(13) in terms of income $Y_t \equiv w_t - \theta_{t-1} E_{t-1} w_t$ and total savings W_t invested in capital $p_{t+1}^k k_{t+1}^m$ with return

$R_{t+1}^k \equiv r_{t+1}^k/p_t^k$ and loans $-b_{t+1}^m$ with return R_{t+1} . Letting α_t be the portfolio weight on capital, the total return on the middle-aged agent's portfolio is $R_{t+1}^m = \alpha_t R_t^k + (1 - \alpha_t) R_{t+1}$ and the budget constraints become

$$\begin{aligned} W_t &= Y_t - c_t^m \\ c_{t+1}^o &= R_{t+1}^m W_t. \end{aligned}$$

The two first-order conditions (14)–(15), in which we substitute $c_{t+1}^o = R_{t+1}^m W_t$, determine the portfolio allocation between bonds and capital: α_t must be such that

$$E_t(R_{t+1}^m)^{-\gamma} R_{t+1} = E_t(R_{t+1}^m)^{-\gamma} R_{t+1}^k$$

In equilibrium the return on capital must satisfy

$$R_{t+1}^k \equiv \frac{r_{t+1}^k}{p_t^k} = \frac{\alpha A_{t+1}^{1-\alpha}}{p_t^k} k_{t+1}^{\alpha-1}, \quad (16)$$

which implies that the risk-free rate and the return on capital are linked by

$$R_{t+1}^k = \frac{\tilde{a}_{t+1}}{\xi_t} R_{t+1} \quad (17)$$

where we have defined

$$\begin{aligned} \xi_t &\equiv \frac{E_t(R_{t+1}^m)^{-\gamma} \tilde{a}_{t+1}}{E_t(R_{t+1}^m)^{-\gamma}} \\ \tilde{a}_{t+1} &\equiv \frac{A_{t+1}^{1-\alpha}}{E_t A_{t+1}^{1-\alpha}}. \end{aligned}$$

The variable \tilde{a}_{t+1} is a transformation of the exogenous shock a_{t+1} .

In addition, the first-order condition (15) yields the saving decision

$$Y_t = \left(1 + (\beta \phi_t R_{t+1}^{1-\rho})^{-\frac{1}{\rho}}\right) W_t \quad (18)$$

where we have defined

$$\phi_t \equiv \left[E_t \left(\left(\frac{R_{t+1}^m}{R_{t+1}} \right)^{1-\gamma} \right) \right]^{(\gamma-\rho)/(1-\gamma)} E_t \left(\left(\frac{R_{t+1}^m}{R_{t+1}} \right)^{-\gamma} \right).$$

We now bring in the market-clearing conditions to express Y_t and W_t in (18) in terms of the aggregate capital stock k_t . First, using the fact that the middle-aged were credit-constrained in their youth, we express their income Y_t as:

$$\begin{aligned} Y_t &= w_t - \theta_{t-1} E_{t-1} w_t \\ &= (1 - \alpha)(\tilde{a}_t - \theta_{t-1}) E_{t-1} A_t^{1-\alpha} k_t^\alpha. \end{aligned} \quad (19)$$

Next, their portfolio choices must equal the supply of the two assets. In (11) the supply of bonds is given by (7) at equality, and the supply of capital is given by (10), which leads to:

$$W_t = p_t^k k_{t+1}^m - b_{t+1}^m = g_{L,t} v_t \frac{p_t^k k_{t+1}}{\alpha \xi_t} \quad (20)$$

where we have defined

$$v_t \equiv \alpha(1 + e^y g_{L,t+1})\xi_t + (1 - \alpha)\theta_t.$$

Similarly we can express $R_{t+1}^m W_t$ as:

$$R_{t+1}^m W_t = R_{t+1}^k p_t^k k_{t+1}^m - R_{t+1} b_{t+1}^m = g_{L,t} u_{t+1} \mathbb{E}_t A_{t+1}^{1-\alpha} k_{t+1}^\alpha \quad (21)$$

where we have defined

$$u_{t+1} \equiv \alpha(1 + e^y g_{L,t+1})\tilde{a}_{t+1} + (1 - \alpha)\theta_t.$$

Taking the ratio of (20) and (21) gives

$$R_{t+1}^m = \frac{u_{t+1}}{v_t} R_{t+1} \quad (22)$$

with u_t and v_{t+1} only function of exogenous variables. Replacing (22) in the definitions of ξ_t and ϕ_t gives:

$$\xi_t = \frac{\mathbb{E}_t(u_{t+1}^{-\gamma} \tilde{a}_{t+1})}{\mathbb{E}_t(u_{t+1}^{-\gamma})} \quad (23)$$

$$\phi_t = [\mathbb{E}_t u_{t+1}^{1-\gamma}]^{(\gamma-\rho)/(1-\gamma)} \mathbb{E}_t u_{t+1}^{-\gamma} v_t^\rho \quad (24)$$

which are also functions of (moments of) the exogenous shock \tilde{a}_{t+1} .

We can now rewrite the middle-aged agent's savings decision (18) as a law of motion for capital by replacing income expressed as (19) and savings expressed as (20):

$$(1 - \alpha)(1 - \frac{\theta_{t-1}}{\tilde{a}_t})\alpha \frac{A_t^{1-\alpha}}{p_t^k} k_t^\alpha = \left(1 + (\beta\phi_t)^{-1/\rho} R_{t+1}^{1-1/\rho}\right) g_{L,t} \frac{v_t}{\xi_t} k_{t+1}. \quad (25)$$

with the left-hand side consisting entirely of variables pre-determined at t .

The expression involves both k and R but (17) allows us to express k_{t+1} in terms of R_{t+1} and vice-versa, so that (25) can be written in terms of the risk-free interest rate R_{t+1} or, equivalently, in terms of the capital stock.

3.3 Discussion

The law of motion (25), which is the core of the model and the basis for our simulations, is the middle-aged agent's optimal choice of saving (18), but with market-clearing imposed on the quantities: the left-hand side represents the savers' income, while the right-hand side is (the inverse of) the saving rate multiplying savings. To develop more intuition we first examine the form it takes in a deterministic steady state, then examine the role of risk.

Deterministic steady state

When we set $\tilde{a}_t = 1$ to shut down the only source of uncertainty the terms involving risk simplify: $\xi_t = 1$ and from (24) $\phi_t = 1$, and $R_t^m = R_t^k = R_t$ (no risk premium).

In steady state (16) implies that $(A_t/k_t)^\alpha/p_t^k$ is constant, hence the growth rate of capital must be $g_k = g_A/g_I^{1/(1-\alpha)}$, as it would be in an infinitely-lived representative agent model. Capital grows at the same rate as labor productivity; the trend in the price of investment goods acts in this respect like an additional form of technological change ($g_I < 1$ leading to growth in the capital stock).

From (19) it also follows that, in steady state, the income of the middle-aged Y_t (and, by (18), their consumption as well as aggregate consumption) grows at the rate $g_I g_k$, that is, the growth rate of capital priced as investment goods. The only determinants of these steady state rates are the technological parameters g_A and g_I . The other parameters affect R and the allocation across generations.

The equation determining the steady state interest rate can be expressed as

$$g_A g_I^{-\frac{1}{1-\alpha}} = (1 + \beta^{-\frac{1}{\rho}} R^{1-\frac{1}{\rho}})^{-1} \left[\frac{1-\alpha}{\alpha g_L} \frac{R}{g_I} \right] \frac{\alpha(1-\theta)}{\alpha(1+e^y g_L) + (1-\alpha)\theta}$$

(see Theorem 1 in Coeurdacier et al. (2015)).

The structure of the equation remains a modified Euler equation. On the left-hand side the term $g_A g_I^{-\frac{1}{1-\alpha}}$ is the steady state rate of growth of capital, which depends only on productivity growth (including the effect of the price in investment goods). This growth rate is unaffected by the various other features of the model. On the right-hand side are three terms. The first is the saving rate. The second term **in square brackets** represents the “pure” OLG component, specifically the fact that those who save do so out of labor income only; capital income is used by the old to finance their consumption. The **last term** captures the effect of the borrowing constraint: this can be seen by setting $e^y = 0$ and $\theta = 0$, which deprives the young of income and prevents them from borrowing, effectively eliminating them. Then that last term reduces to 1, and the model is isomorphic to a two-period overlapping generations model with no borrowing constraint.

Risky steady state

We assume that uncertainty on the productivity can be modelled as an i.i.d process.

Assumption 1 (Distribution of the productivity shock). *Assume that the productivity shock is i.i.d, with mean 1.*

To account for the impact of risk while retaining tractability, we appeal to the concept of risky steady state (Juillard 2011, Coeurdacier et al. 2011). Instead of setting $\tilde{a}_t = \tilde{a}_{t+1} = 1$ as in the deterministic steady state, we set \tilde{a}_t to its mean of 1 but maintain \tilde{a}_{t+1} as a stochastic variable, assuming that it is lognormal with variance σ_{t+1} . In effect, we assume that in every period the current realization of the shock is at its mean but agents take into account the risk in the next period.

The following result describes the behavior of the risky steady-state.

Proposition 1. *The risky steady-state satisfies the following equation*

$$g_A g_I^{-\frac{1}{1-\alpha}} = (1 + (\phi\beta)^{-\frac{1}{\rho}} R^{1-\frac{1}{\rho}})^{-1} \left[\frac{1-\alpha}{\alpha g_L} \frac{R}{g_I} \right] \frac{\alpha(1-\theta)}{\alpha(1+e^y g_L)\xi + (1-\alpha)\theta} \quad (26)$$

This equation admits a unique solution.

Compared to the deterministic case, the presence of risk adds two channels, captured by the terms ϕ and ξ , functions of the exogenous factors only.⁴

The first channel ξ relates to the risk premium and its impact on the portfolio choice of the agent. This can be seen in two ways. First, from (17) the risk premium R^k/R is \tilde{a}_{t+1}/ξ_t . Second, the share of the agent's savings invested in capital (the risky asset) is $\alpha(1+e^y g_{L,t+1})\xi_t/v_t$, which is proportional to the term $\xi_t/v_t = \xi_t/(\alpha(1+e^y g_L)\xi_t + (1-\alpha)\theta)$.

The following result describes the properties of the risky steady-state, when \tilde{a} follows a log-normal law.

Proposition 2. 1. *The risk-premium ξ^{-1} admits the following asymptotic expansion*

$$\xi^{-1}(\theta, g_L, \sigma) = 1 + \gamma \frac{\alpha(1+e^y g_L)}{(1-\alpha)\theta + \alpha(1+e^y g_L)} \sigma^2 + o(\sigma^4)$$

2. *The distortion ϕ admits the following asymptotic expansion*

$$\phi_t = 1 + \frac{1}{2} \gamma (1-\rho) \frac{\alpha^2(1+e^y g_{L,t+1})^2}{(\alpha(1+e^y g_{L,t+1}) + (1-\alpha)\theta_t)^2} \sigma^2 + o(\sigma^4).$$

3. *The portfolio-share of middle-aged allocated to capital is given by*

$$\frac{\alpha(1+e^y g_{L,t+1})\xi_t}{\alpha(1+e^y g_{L,t+1})\xi_t + (1-\alpha)\theta_t}$$

The risk premium obviously increases with risk aversion γ , but also with a tightening of the borrowing constraint (lower θ) or a fall in population growth g_L , both of which reduce the supply of bonds. These effects, however, are second-order only: there are no first-order terms for θ or g_L .

The second channel, ϕ_t , acts like a distortion to the discount rate, and is familiar from the literature on recursive preferences.

There are two things to note. One is that the sign of the sensitivity of ϕ_t to risk depends on whether the intertemporal elasticity ρ is high or low relative to 1. The discussion in Weil (1990, 38) applies here: a high IES ($\rho < 1$) means that the income effect is small relative to the substitution effect, and the “effective” discount factor $\phi\beta$ rises with risk: the agent behaves as if she were more patient, and a higher interest rate R is required in equilibrium. The IES determines the sign, but the magnitude of the effect is determined by the risk aversion γ .

The second point is that the relative strength of the two channels (the ratio of $\partial\phi/\partial\sigma^2$ to $\partial\xi/\partial\sigma^2$) is $(1-\rho)\alpha(1+e^y g_L)/2(\alpha(1+e^y g_L) + (1-\alpha)\theta)$ which, for low θ , is close to

⁴When $\delta \neq 1$ they are also functions of the endogenous rates of return (see the appendix).

$(1 - \rho)/2$. For a IES close to 1, the effect of risk through the precautionary channel will be much smaller (in our calibration of $\rho = 0.8$, one order of magnitude smaller) than through the portfolio channel. For log utility (IES=1) there is only the portfolio channel.

When $\rho = 1$ we find the following first-order approximation for R and R^k around $[g_I, g_A, g_L, \theta, \sigma^2] = [1, 1, 1, 0, 0]$:

$$\begin{aligned}\ln(R) &= \bar{r} + \frac{1 + 2e^y}{1 + e^y} \ln(g_L) + \ln(g_A) - \frac{\alpha}{1 - \alpha} \ln(g_I) + \frac{1 + \alpha e^y}{\alpha(1 + e^y)} \theta - \gamma \sigma^2 \\ \ln(R^k) &= \bar{r} + \frac{1 + 2e^y}{1 + e^y} \ln(g_L) + \ln(g_A) - \frac{\alpha}{1 - \alpha} \ln(g_I) + \frac{1 + \alpha e^y}{\alpha(1 + e^y)} \theta\end{aligned}$$

with

$$\bar{r} = \ln \left[\frac{\alpha(1 + e^y)(1 + \beta)}{(1 - \alpha)\beta} \right]$$

Thus, even for $\rho = 1$ there is room for risk to affect interest rates, through the portfolio channel. The return to capital, however, does not depend on risk. Increasing risk raises the risk premium and compresses the risk-free rate, which is (roughly) what we see in the data. Indeed, risk is the only one of our “suspects” that affects only the risk-free rate.

4 A Quantitative Evaluation

In this section we use the model to match the data on the risk free real interest rate and the return of capital since 1970.

4.1 Calibration of the model

The spirit of the exercise is as follows.

We distinguish between (a) the structural parameters $\beta, \gamma, \rho, \alpha, \delta, e^y$ characterizing preferences and technology, held fixed throughout and calibrated in a standard way, and (b) the factors g_I, g_A, g_L that vary over time but are readily observable (Table 1).

This leaves us with θ (the borrowing constraint) and σ (the amount of risk) which we approach flexibly because we see them as less easily observable. We proceed in several steps. First, we fix both $\theta = 0.07$ and $\sigma = 0.09$ for the US and the EA to see how much the observable factors can explain.⁵ Then we keep one fixed and compute a time series for the other in order to match the path of the risk-free rate R . Finally let both vary and we back out time series of θ_t and σ_t that will result in sequences of risky steady states R and R^k matching the observations. Ultimately, of course, we will need to confront these time series to data in order to assess the model’s success or failure at accounting for the decline in interest rates.

Our model periods last 10 years. In the figures that follow, each year N on the x -axis corresponds to the average 10-year lagging average (years $N - 9$ to N), both for data and

⁵These calibrations are chosen roughly in terms of the estimated values on the whole period, the values have an incidence on the level of the interest rates, but they impact marginally their evolution

<i>Parameters</i>		
T	length of period (years)	10
β	discount factor	0.98^T
α	capital share	0.28
γ	risk aversion	100
ρ	IES	0.8
δ	capital depreciation rate	$0.1 * T$
e^y	relative productivity of young	0.3
<i>Factors</i>		
$g_{L,t}$	growth rate of population 20-64	US, EA (France), China, Japan: OECD
$g_{I,t}$	investment price growth	DiCecio (2009)
$g_{A,t}$	productivity growth	US: Fernald (2012), Euro: NAWM model
R_t	real interest rate	US: Hamilton et al. (2016), France
R_t^k	return on capital	US, EA: our calculations à la
		Gomme et al. (2015)
\tilde{a}_t	productivity shock	$\ln(\tilde{a})$ is a i.i.d. $N(-\sigma^2/2, \sigma^2)$
<i>Free parameters</i>		
θ	borrowing constraint on young	
σ^2	variance of \tilde{a}_t	

Table 1: Model calibration and data sources.

simulations.⁶ Our reasoning is that deciding when our 10-year periods start and end is somewhat arbitrary. Presenting the data and simulations in this manner avoids making that decision, as long as the reader keeps in mind that we are not representing annual time series, but sequences of $\{t, t + 10\}$ pairs.

4.2 Results

The impact of observable factors

To measure this impact we fix θ and σ and analyze the model-based interest rates, when we use as inputs the growth rate of productivity, the change in demography and in relative investment price that we observe in the data (Figures 4 and 5). These factors are represented in Figure 3.⁷ Combined, these inputs reduce both the risk free rate and the return on capital by about 0.7 percent from 1990 to 2014 with no effect on the risk premia for the US, by 2.3 percent for the EA on the same period. These estimates are comparable to the ones of Gagnon et al. (2016) and Carvalho et al. (2016). The former estimate that demography account for a decline of the US equilibrium real rate by 1.25 percent from 1980 to 2015. And the latter estimate a decline by 1.5 percent between 1990 and 2014.

The borrowing constraint

To measure the explanatory power of the borrowing constraint, we fix σ and compute the parameter θ which is consistent with the risk-free rate, and the observable inputs. The

⁶At the beginning of the sample, we compute the average on the available data, which starts in 1960, except for the productivity for the EA which starts in 1970

⁷The risk premia in the euro area appears very high in the 1970s again because at that period, financial repression implies very low real interest rates. For the euro area, we believe that the data are more meaningful for the post 1985 period as capital controls are progressively removed in Europe

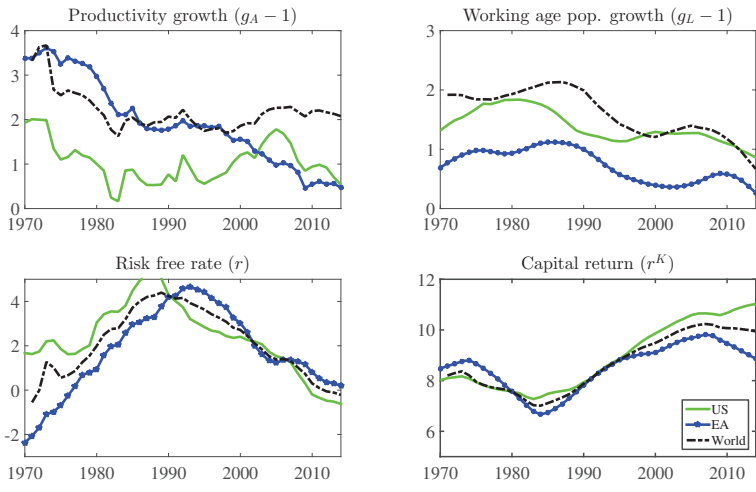


Figure 3: The inputs

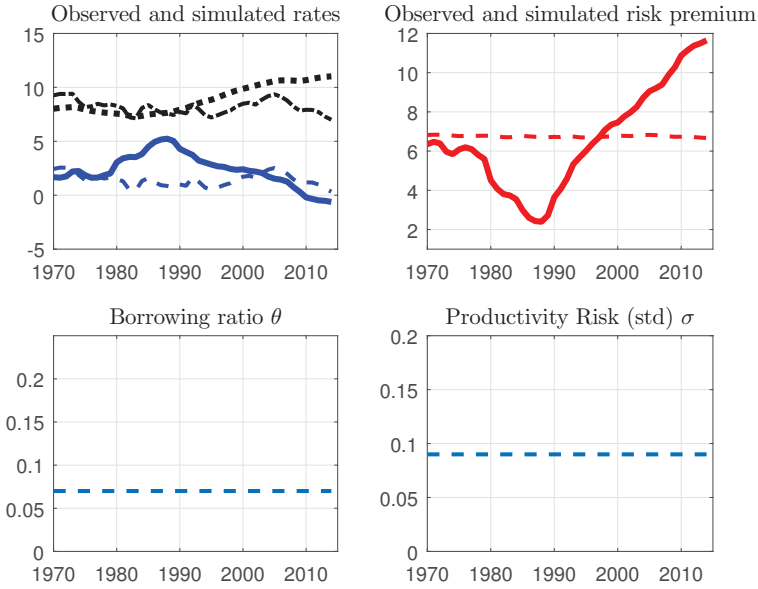


Figure 4: Impact of observable factors, in the US.

implied borrowing constraints and the model-based return on capital, and risk-premia are represented in Figures 6, for the US and 7, for the EA. This exercise shows that for both areas, the decline in the risk-free rate requires a tightening in the borrowing constraint, from 0.15 in 1990 to 0.05 in the US (from 0.12 to 0.08 in the EA). This evolution of the borrowing constraint hardly replicates the increase in the risk-premium in both areas, 1.1% in the US between 1990 and 2014 (0.2% in the EA on the same period).

Risk

We now consider the borrowing constraint as fixed over time, and assess the evolution of the variability in productivity that is required to reproduce the decline in the risk-free rate.

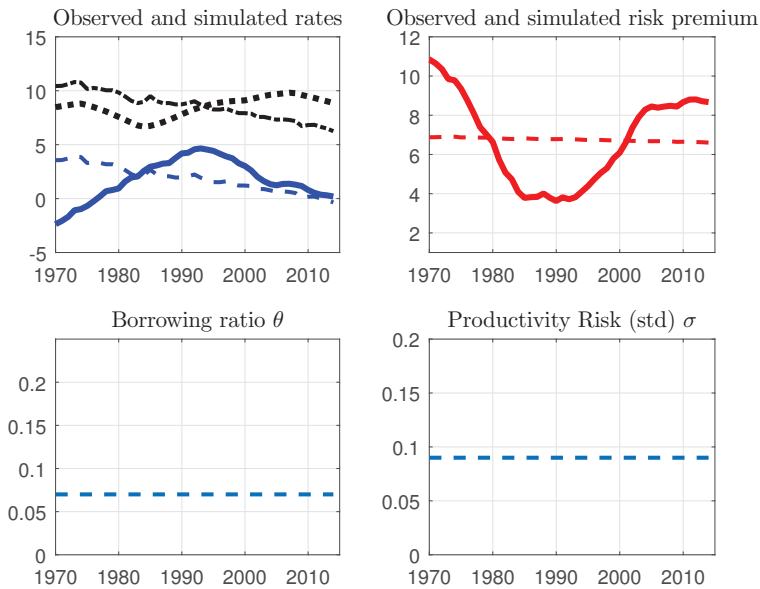


Figure 5: Impact of observable factors, in the EA.

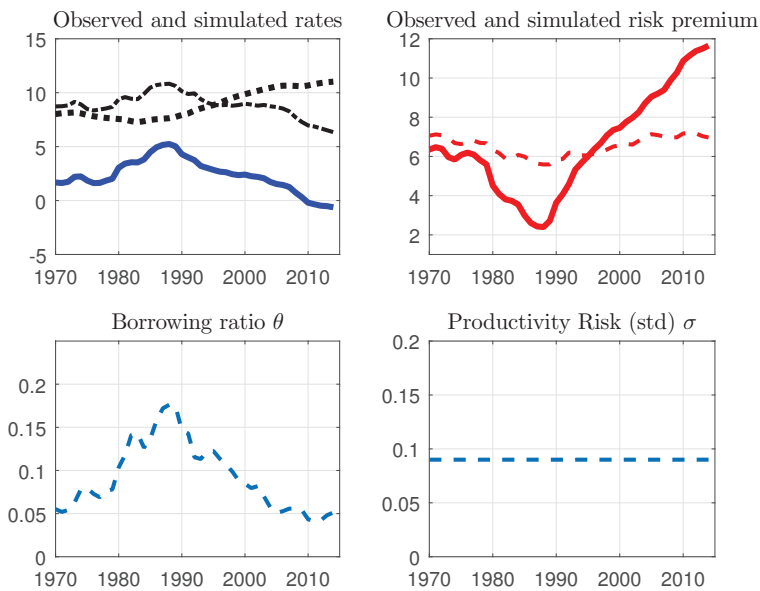


Figure 6: Impact of the borrowing constraint, in the US.

The implied change in variability and the model-based return on capital are represented in Figures 8, for the US and 9, for the EA. The variance of productivity has to increase from about 0.04 per cent in 1990 to 0.1 today, for the US (from 0.06 to 0.09 for the EA). For both areas, this evolution since 1990 replicates quite well the evolution in the return on capital, and risk-premium.

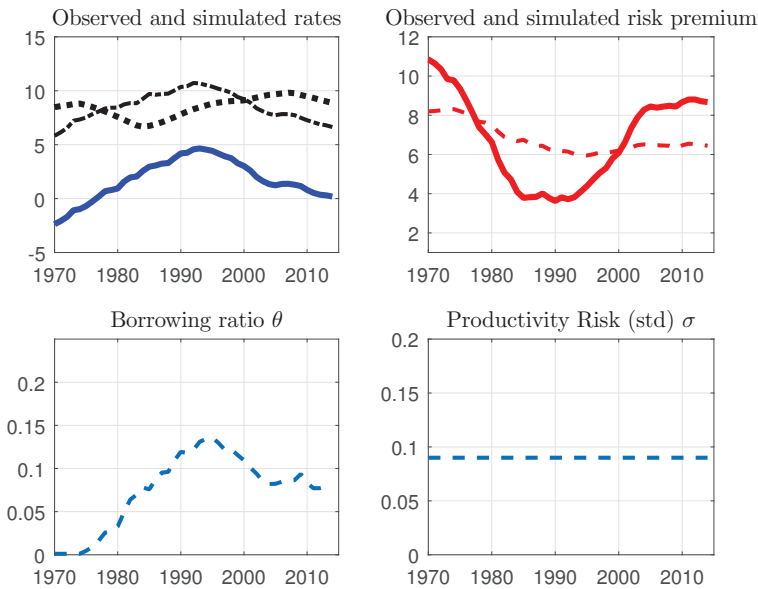


Figure 7: Impact of the borrowing constraint, in the EA.

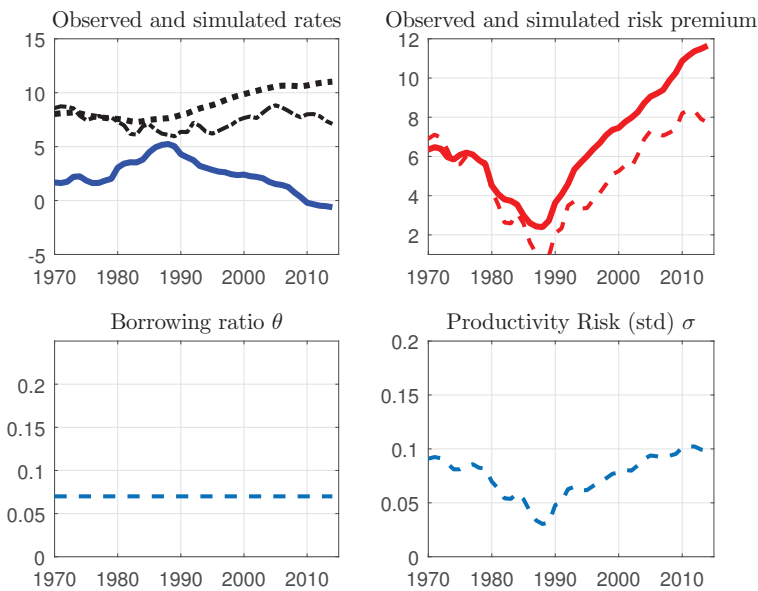


Figure 8: Impact of risk, in the US.

Risk and the borrowing constraint

In Figures 10, for the US and 11, for the EA, we let both θ and σ change over time so that we can replicate perfectly both the risk free rate and the return on capital. In particular, the trend decline in the risk free rate since 1990 is due exclusively to an increase in the risk of productivity, from 0.06 to 0.14 in the US (0.06 to 0.11 in the EA). Moreover, the evolution of the risk-free rate and the return on capital is consistent with a non decreasing pattern of the borrowing constraint. This shows that, according to the model, the drop in real interest rate need not reflect deleveraging headwinds. What evidence do we have that

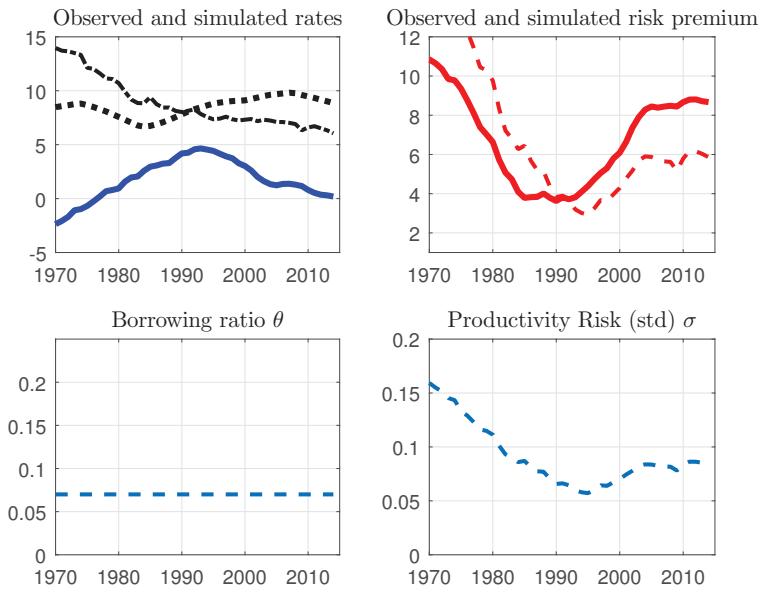


Figure 9: Impact of risk, in the EA.

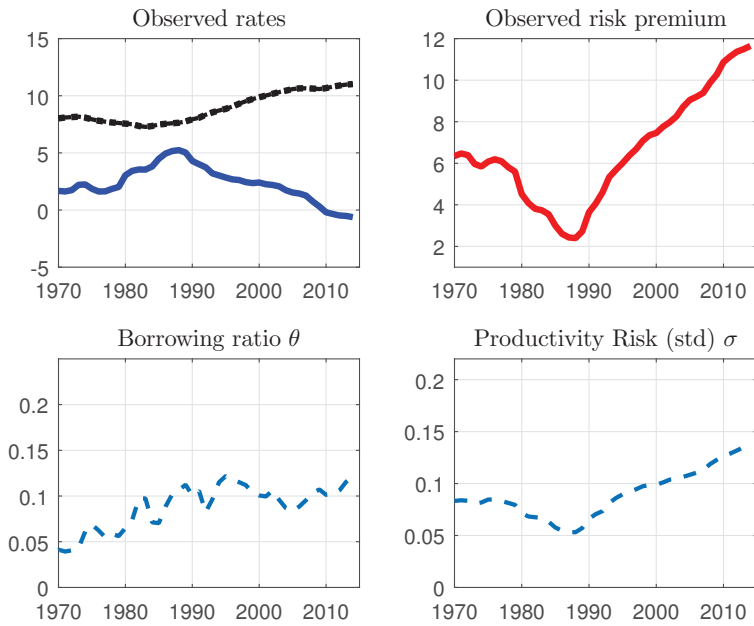


Figure 10: Impact of risk and the borrowing constraint, in the US.

uncertainty as effectively increased over the last 25 years? Baker et al. (2016) indicates that there may be an upward trend of economic uncertainty from the 1985 to 2012 and their a clear acceleration of political uncertainty over the last fifteen years. In particular the so called "great moderation" period, usually dated from 1985 to 2007 does not correspond to a decline in uncertainty as measured by Baker et al. (2016). Altogether, that our simulation point to an increase in perceived risk as suggested by the steady increase in the risk premia from 1990 to 2016 is not inconsistent with these other measures that show uncertainty

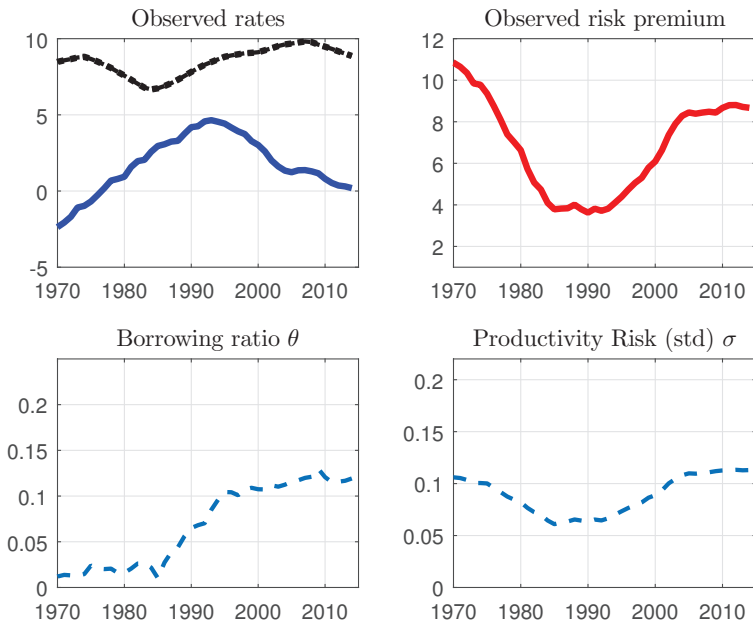


Figure 11: Impact of risk and the borrowing constraint, in the EA.

trending up, at least from 1990 to 2012.

The evolution of the borrowing constraint ?

As shown by Buttiglione et al. (2014) there has been hardly any overall deleveraging since the crisis. Private debt in advanced economies adds up to 178 per cent of GDP in 2016, i.e. the same level than in 2010, while public debt increased from 75 to 87 per cent of GDP over the same period. Deleveraging of the private sector has been very large in Spain and in the United Kingdom, but it increased in France and Canada. And public debt increased in all G7 countries but Germany.

We thus use the model to infer the borrowing constraint consistent with the evolution of debt and the risk-free rate in both areas, depicted in Figures 12, for the US and 13, for the EA. In this exercise, we compute the borrowing constraint and the level of risk implied by the model to replicate the ratio debt/income⁸ and the risk-free rate. This shows that the pattern of the borrowing constraint consistent with the evolution of the debt implies an increase in variability of productivity completely in line with what we observe in the US to replicate the risk-premium, a bit smaller in the EA.

4.3 A global perspective

A fair criticism of our calculations is that, by focusing on the US and the Euro area, we neglect the phenomenon described as “savings glut” or “global imbalances” of the 2000s,

⁸We consider that a proxy of this ratio is the total debt over T times the GDP

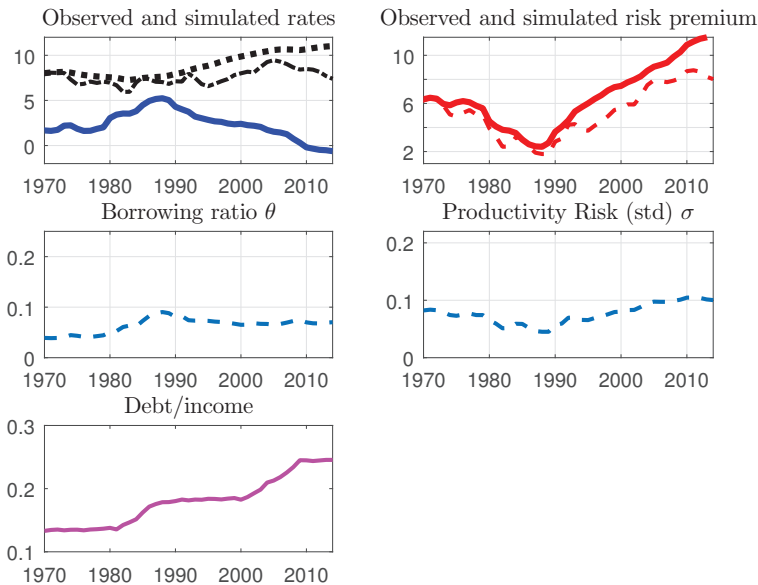


Figure 12: Borrowing constraint and risk, in the US.

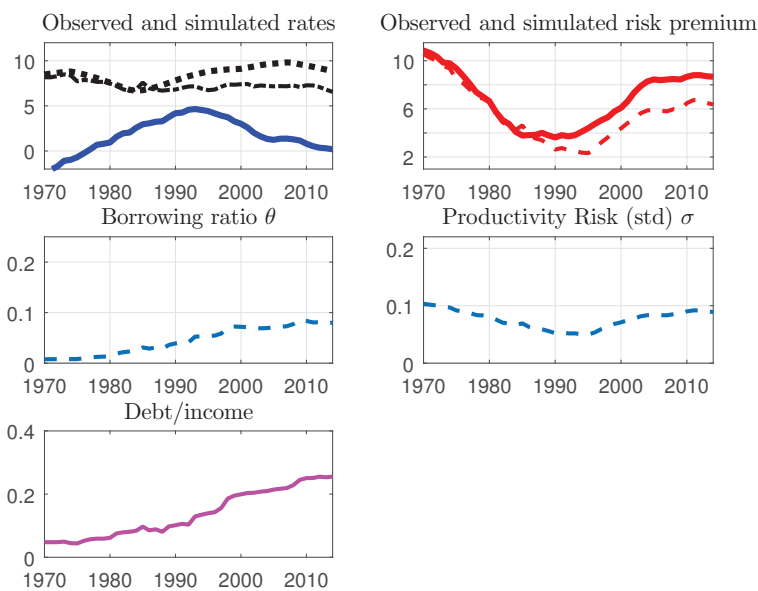


Figure 13: Borrowing constraint and risk, in the EA.

namely the increase in savings from emerging economies. We repeat our calculations for the world by adding Japan and China. We consider that a representation of the world is described by the aggregation of EA, US, China and Japan. The "world" population aged 20 to 64 is the ones of these four economies, investment price evolves as the American one, whole productivity is described as an aggregate of four productivities weighted by GDP. The target rates, both for return on capital and for the risk-free rate, are an average of the US and Euro area, considering that world capital markets are integrated. We think that our calculations are most likely to shed light on the period since 1990, once financial

repression mechanisms have dropped out of the picture for most economies.

The results at the world level are broadly similar to the ones we found for the US and the EA: risk is the main factor that can account for the behavior of the risk premium since 1990. We note that, in the simulation that lets θ vary, in the early years the parameter θ is constrained at zero (which is why we cannot match the return on capital perfectly. The picture is nevertheless similar: we see θ rising since the mid-1980s, suggestive of the global savings glut. the borrowing ratio stops rising in the late 1990s and barely falls after that: deleveraging does not seem to be at play since the financial crisis.

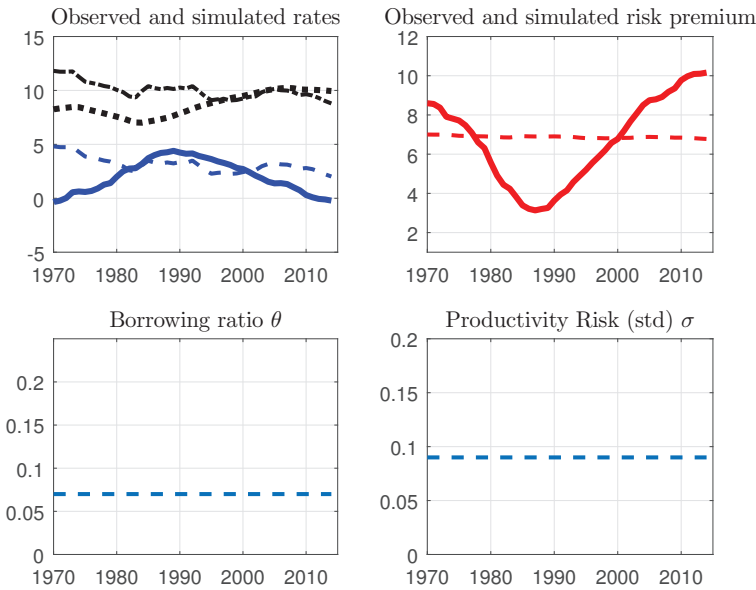


Figure 14: Impact of observable factors, world.

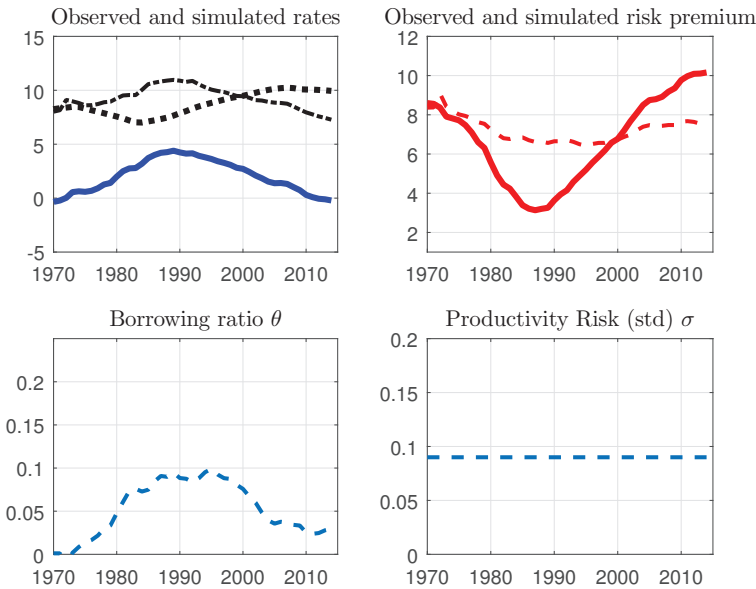


Figure 15: Impact of the borrowing constraint, world.

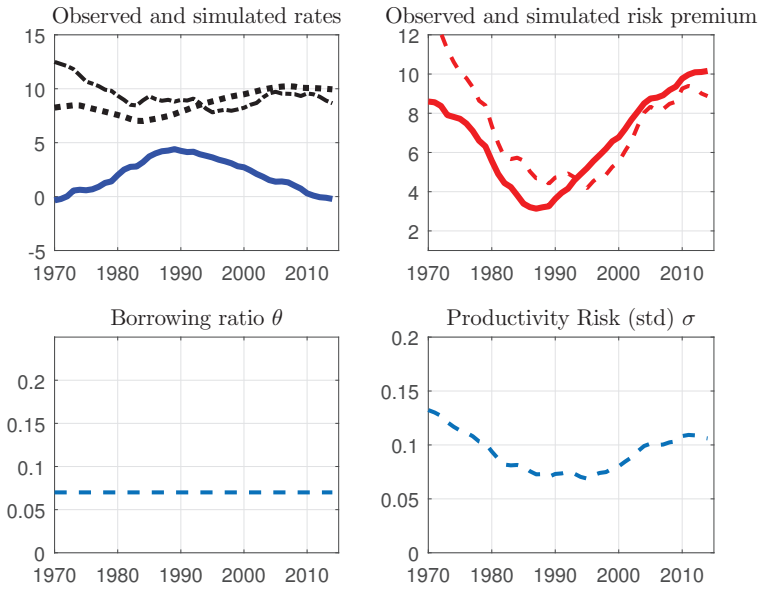


Figure 16: Impact of risk, world

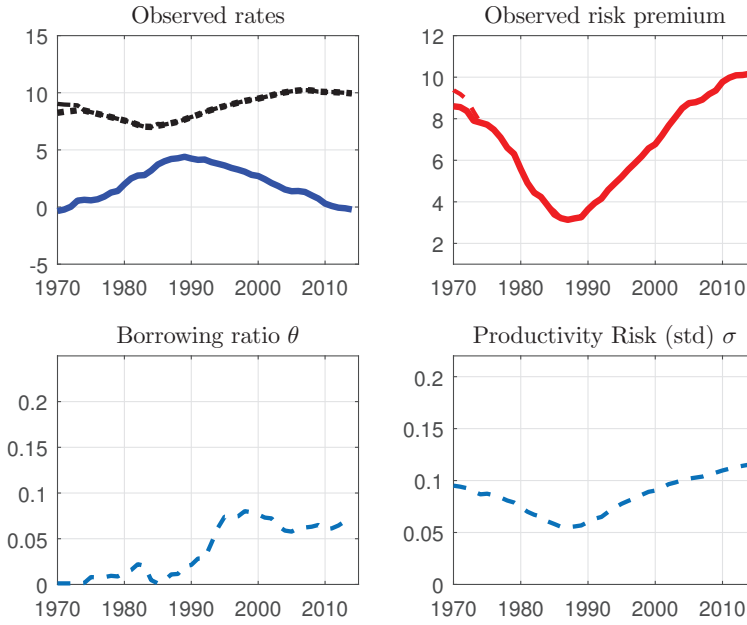


Figure 17: Impact of risk and the borrowing constraint, world.

5 Conclusion

Risk-free rates have been falling since the 1980s while the return on capital has not. We analyze these trends in a calibrated overlapping generations model designed to encompass many of the "usual suspects" cited in the debate on secular stagnation. Declining labor force and productivity growth imply a limited decline in real interest rates and deleveraging cannot account for the joint decline in the risk free rate and increase in the risk premium. If we allow for a change in the (perceived) risk to productivity growth to fit the

data, we find that the decline in the risk-free rate requires an increase in the borrowing capacity of the indebted agents in the model, consistent with the increase in the sum of public and private debt since the crisis but at odds with a deleveraging-based explanation put forth in Eggertsson and Krugman (2012).

Appendix

General case ($\delta \neq 1$)

It will be convenient to denote the return to investment expressed in terms of consumption as z_{t+1} :

$$z_{t+1} \equiv \frac{p_t^k}{p_{t+1}^k} R_{t+1}^k - 1 + \delta = \alpha \frac{A_{t+1}^{1-\alpha}}{p_{t+1}^k} k_{t+1}^{\alpha-1}$$

Equation (17) becomes:

$$\begin{aligned} E_t(R_{t+1}^{m_{t+1}^{-\gamma}}) R_{t+1} &= E_t(R_{t+1}^{m_{t+1}^{-\gamma}} R_{t+1}^k) \\ R_{t+1} &= \xi_t \alpha \frac{E_t A_{t+1}^{1-\alpha} k_{t+1}^\alpha}{p_t^k k_{t+1}} + (1-\delta) \frac{p_{t+1}^k}{p_t^k} \end{aligned} \quad (27)$$

$$= \xi_t \frac{\alpha}{1-\alpha} \frac{E_t w_{t+1}}{p_t^k k_{t+1}} + (1-\delta) \frac{p_{t+1}^k}{p_t^k} \quad (28)$$

where we have defined

$$\begin{aligned} \xi_t &\equiv \frac{E_t(R_{t+1}^{m_{t+1}^{-\gamma}} \tilde{a}_{t+1})}{E_t R_{t+1}^{m_{t+1}^{-\gamma}}} \\ \tilde{a}_{t+1} &\equiv \frac{A_{t+1}^{1-\alpha}}{\mathbb{E}_t A_{t+1}^{1-\alpha}}. \end{aligned}$$

Note that the risk-free rate and the return on capital are now linked as follows:

$$R_{t+1}^k = \frac{\tilde{a}_{t+1}}{\xi_t} R_{t+1} + (1 - \frac{\tilde{a}_{t+1}}{\xi_t}) \frac{p_{t+1}^k}{p_t^k} (1 - \delta) \quad (29)$$

The auxiliary variables v_t and u_{t+1} take the more general form

$$\begin{aligned} v_t &\equiv \alpha(1 + e^y g_{L,t+1}) \xi_t + (1-\alpha) \theta_t \left(1 - \frac{p_{t+1}^k}{p_t^k} \frac{1-\delta}{R_{t+1}}\right) \\ &= \alpha(1 + e^y g_{L,t+1}) \xi_t + (1-\alpha) \theta_t \frac{\tilde{z}_{t+1}}{\tilde{z}_{t+1} + 1 - \delta} \\ u_{t+1} &\equiv \alpha(1 + e^y g_{L,t+1}) (\tilde{a}_{t+1} + p_{t+1}^k (1-\delta) \frac{k_{t+1}^{1-\alpha}}{\alpha \mathbb{E}_t A_{t+1}^{1-\alpha}}) + (1-\alpha) \theta_t \\ &= \alpha(1 + e^y g_{L,t+1}) (\tilde{a}_{t+1} + \frac{\xi_t}{z_{t+1}} (1-\delta)) + (1-\alpha) \theta_t \end{aligned}$$

and R_{t+1}^m becomes

$$\begin{aligned} R_{t+1}^m &= \frac{u_{t+1}}{v_t} \frac{\alpha \xi_t}{1-\alpha} \frac{(1-\alpha) \mathbb{E}_t A_{t+1}^{1-\alpha} k_{t+1}^\alpha}{p_t^k k_{t+1}} \\ &= \frac{u_{t+1}}{v_t} \frac{\alpha \xi_t}{1-\alpha} \frac{\mathbb{E}_t w_{t+1}}{p_t^k k_{t+1}} \\ &= \frac{u_{t+1}}{v_t} \left(R_{t+1} - \frac{p_{t+1}^k}{p_t^k} (1-\delta) \right) = \frac{u_{t+1}}{v_t} \frac{p_{t+1}^k}{p_t^k} \frac{\xi_t}{\tilde{a}_{t+1}} z_{t+1}. \end{aligned} \quad (30)$$

Replacing (30) in the definitions of ξ_t and ϕ_t gives:

$$\xi_t = \frac{\mathbb{E}_t(u_{t+1}^{-\gamma} \tilde{a}_{t+1})}{\mathbb{E}_t(u_{t+1}^{-\gamma})} \quad (31)$$

$$\phi_t = [\mathbb{E}_t u_{t+1}^{1-\gamma}]^{(\gamma-\rho)/(1-\gamma)} \mathbb{E}_t u_{t+1}^{-\gamma} v_t^\rho \left(1 - \frac{p_{t+1}^k}{p_t^k} \frac{1-\delta}{R_{t+1}}\right)^{-\rho} \quad (32)$$

$$= [\mathbb{E}_t u_{t+1}^{1-\gamma}]^{(\gamma-\rho)/(1-\gamma)} \mathbb{E}_t u_{t+1}^{-\gamma} (u_{t+1} + \alpha(1 + e^y g_{L,t+1})(\xi_t - \tilde{a}_{t+1}))^{-\rho} \quad (33)$$

The general form of the law of motion is

$$(1 - \alpha)(1 - \frac{\theta_{t-1}}{\tilde{a}_t})z_t k_t = \left(1 + (\beta\phi_t)^{-1/\rho} R_{t+1}^{1-1/\rho}\right) g_{L,t} \frac{v_t}{\xi_t} k_{t+1}. \quad (34)$$

with the left-hand side consisting entirely of variables pre-determined at t .

To compute the risky steady state, we first express the law of motion (34) in terms of R_{t+1} , we proceed as follows. First, we replace z_t with $\tilde{z}_t \tilde{a}_t / \xi_{t-1}$ to write (34) as:

$$(1 - \alpha)(\tilde{a}_t - \theta_{t-1})\tilde{z}_t = \left(1 + (\beta\phi_t)^{-1/\rho} R_{t+1}^{1-1/\rho}\right) g_{L,t} v_t \frac{\xi_{t-1}}{\xi_t} \frac{k_{t+1}}{k_t}.$$

Then we use (27) to rewrite k_{t+1}/k_t :

$$\begin{aligned} \frac{R_{t+1} - (1 - \delta)g_{I,t+1}}{R_t - (1 - \delta)g_{I,t}} &= \frac{\xi_{t+1}}{\xi_t} g_{I,t+1} \frac{\mathbb{E}_t A_{t+1}^{1-\alpha}}{\mathbb{E}_{t-1} A_t^{1-\alpha}} \left(\frac{k_{t+1}}{k_t}\right)^{\alpha-1} \\ &= g_{I,t+1} \frac{\xi_{t+1}}{\xi_t} g_{A,t+1}^{1-\alpha} \left(\frac{k_{t+1}}{k_t}\right)^{\alpha-1} \\ \frac{k_{t+1}}{k_t} &= g_{I,t+1}^{1/(1-\alpha)} \left(\frac{\xi_t}{\xi_{t-1}}\right)^{1/(1-\alpha)} g_{A,t+1} \left(\frac{R_t - (1 - \delta)g_{I,t}}{R_{t+1} - (1 - \delta)g_{I,t+1}}\right)^{1/(1-\alpha)} \end{aligned}$$

so that

$$(1 - \alpha)(\tilde{a}_t - \theta_{t-1}) \left[\frac{R_t - (1 - \delta)g_{I,t}}{g_{I,t}} \right] = \left[1 + (\beta\phi_t)^{-1/\rho} R_{t+1}^{1-1/\rho} \right] g_{L,t} \times \left[\alpha(1 + e^y g_{L,t+1})\xi_t + (1 - \alpha)\theta_t(1 - g_{I,t+1} \frac{1 - \delta}{R_{t+1}}) \right] (g_{I,t+1})^{1/(1-\alpha)} \left(\frac{\xi_t}{\xi_{t-1}}\right)^{\alpha/(1-\alpha)} \quad (35)$$

$$\times g_{A,t+1} \left(\frac{R_t - (1 - \delta)g_{I,t}}{R_{t+1} - (1 - \delta)g_{I,t+1}}\right)^{1/(1-\alpha)}. \quad (36)$$

This form of the law of motion involves additional variables ξ_t and ϕ_t satisfying (31– 32) with

$$u_{t+1} = \alpha(1 + e^y g_{L,t+1}) \left(\tilde{a}_{t+1} + \frac{g_{I,t+1}(1 - \delta)\xi_t}{[R_{t+1} - (1 - \delta)g_{I,t+1}]} \right) + (1 - \alpha)\theta_t. \quad (37)$$

Equation (36), along with (31), (32), and (37), define (R_{t+1}, ξ_t) implicitly as a recursive function of (R_t, ξ_{t-1}) and \tilde{a}_t :

$$\begin{bmatrix} R_{t+1} \\ \xi_t \end{bmatrix} = f\left(\begin{bmatrix} R_t \\ \xi_{t-1} \end{bmatrix}, \tilde{a}_t, \tilde{a}_{t+1}\right). \quad (38)$$

Note that the arguments of (38) include the realization of \tilde{a}_t (known at t) and the (conditional) probability distribution of \tilde{a}_{t+1} , which is the only source of uncertainty in the model.

We define the risky steady-state as satisfying the relation

$$\begin{bmatrix} \bar{R} \\ \bar{\xi} \end{bmatrix} = f\left(\begin{bmatrix} \bar{R} \\ \bar{\xi} \end{bmatrix}, 1, \tilde{a}_{t+1}\right)$$

which leads to

$$\begin{aligned} (1 - \alpha)(1 - \theta)(\bar{R}/g_I - 1 + \delta) &= (1 + \beta^{-1/\rho} R^{1-1/\rho}) g_L g_A g_I^{-1/(1-\alpha)} \\ &\times \left[\alpha(1 + e^y g_L)\bar{\xi} + (1 - \alpha)\theta \left(1 - (1 - \delta)\frac{g_I}{\bar{R}}\right) \right] \\ \bar{\xi} &= \xi(\bar{R}, \bar{\xi}). \end{aligned}$$

The equation determining the deterministic steady state interest rate can be expressed as

$$\frac{g_A}{g_I^{\frac{1}{1-\alpha}}} = (1 + \beta^{-\frac{1}{\rho}} R^{1-\frac{1}{\rho}})^{-1} \left[\frac{1-\alpha}{\alpha g_L} \left(\frac{R}{g_I} - 1 + \delta \right) \right] \frac{\alpha(1-\theta)}{\alpha(1+e^y g_L) + (1-\alpha)\theta(1-g_I^{\frac{1-\delta}{R}})}$$

Proof of Proposition 1

We give the details of the proof of Proposition 1. The proof is in two steps. First, we establish the dynamic equation of R_t , to obtain the equation at the steady-state. Then, we study the properties of R as a function of the inputs.

Dynamic relation

Starting from equation (25), the dynamic relation is rewritten as

$$(1-\alpha)(1 - \frac{\theta_{t-1}}{\tilde{a}_t}) \alpha \frac{A_t^{1-\alpha}}{p_t^k} k_t^\alpha = \left(1 + (\beta \phi_t)^{-1/\rho} R_{t+1}^{1-1/\rho} \right) g_{L,t} \frac{v_t}{\xi_t} k_{t+1}$$

This leads to the following law of motion for R_t

$$(1-\alpha)R_t(\tilde{a}_t - \theta_{t-1}) = \left(1 + (\beta \phi_t)^{-1/\rho} R_{t+1}^{1-1/\rho} \right) g_{L,t} g_{I,t}^{-\alpha/(1-\alpha)} g_{A,t} \left(\frac{\xi_t}{\xi_{t-1}} \right)^{\alpha/(1-\alpha)} v_t \left(\frac{R_{t+1}}{R_t} \right)^{-1/(1-\alpha)} \quad (39)$$

Equation at the steady-state, for $\delta = 1$

We introduce

$$M(p) = \int [\alpha(1 + e^y g_L) \tilde{a} + (1-\alpha)\theta]^{-p} d\tilde{a}$$

We compute

$$\alpha(1 + e^y g_L) \xi + (1-\alpha)\theta = \frac{M(\gamma-1)}{M(\gamma)}$$

$$\phi = M(\gamma-1)^{\rho+(\gamma-\rho)/(1-\gamma)} M(\gamma)^{1-\rho} = \left(M(\gamma-1)^{\gamma/(\gamma-1)} M(\gamma)^{-1} \right)^{\rho-1}$$

R is solution of the equation

$$\frac{R}{1 + \beta^{-1/\rho} [M(\gamma-1)^{\gamma/(\gamma-1)} M(\gamma) R]^{1-1/\rho}} = \frac{g_L g_A g_I^{-\alpha/(1-\alpha)} M(\gamma-1)}{(1-\alpha)(1-\theta) M(\gamma)}$$

The proof mimics the proof of Theorem 1 in Coeurdacier et al. (2015), we consider the function

$$h(R) = \frac{R}{1 + \beta^{-1/\rho} [M(\gamma-1)^{\gamma/(\gamma-1)} M(\gamma) R]^{1-1/\rho}}$$

h is strictly increasing, for $\rho \leq 1$, it defines a bijection from $[0, +\infty)$ to $[0, +\infty)$, thus there exists a unique R such that

$$h(R) = \frac{g_L g_A g_I^{-\alpha/(1-\alpha)} M(\gamma-1)}{(1-\alpha)(1-\theta) M(\gamma)}$$

The directions of variation of R with g_A , g_I and β are obvious.

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