

Sraffa and von Neumann

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This paper discusses the relationship between Piero Sraffa's 1960 book and John von Neumann's 1937 paper on economic growth in the light of some of the material contained in Sraffa's unpublished papers and correspondence. It is argued that the two contributions share a similar outlook and exhibit conceptual parallels; in fact, they can both be said to belong to the 'classical' approach to the theory of value and distribution. The latter is characterized, among other things, by an asymmetric treatment of the distributive variables, the rate of return on capital and the real wage rate. Sraffa's papers show that when he came across the von Neumann model in the mid-1940s his own analysis was already quite advanced, including his analysis of joint production. The paper also contains an exchange of letters between John Richard Hicks and Sraffa on some of the issues dealt with in the latter's book.

1. Introduction

The relationship between Piero Sraffa's (1960) *Production of Commodities by Means of Commodities* and John von Neumann's ([1937] 1945) paper on economic growth has given rise to various assessments and comments. This is understandable, because the analyses presented by the two authors exhibit several similarities. In particular, they use a similar method of analysis, that is, they are concerned with long-period positions of a competitive economic system characterized by a uniform rate of return; and they study the problem of prices, distribution and the choice of technique, in an intersectoral framework in which production is conceived of as a circular flow. A feature of both contributions is that the distributive variables, the real wage rate and the rate of interest (or rate of profits)¹, are treated asymmetrically: one of these variables is given from outside the system of production, while the other is determined as a residual. Fixed capital is dealt with in a joint products framework. Despite these similarities and common concerns the assessments of the relationship between the two authors differ vastly across different interpretations. While some com-

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¹ Von Neumann uses the term 'rate of interest', whereas Sraffa uses the term 'rate of profits'. However, as will become clear below, they mean essentially the same thing. Therefore, the two terms will be considered as synonymous in this paper.

mentators argue that the analyses of Sraffa and von Neumann are broadly compatible with one another and can be shown to benefit from each other, others maintain that they belong to different traditions of economic thought and are characterized by conceptual incompatibilities.

Von Neumann's paper on economic growth was originally published in German in 1937 in Karl Menger's *Ergebnisse eines mathematischen Kolloquiums* and then, on the initiative of Nicholas Kaldor, translated into English and published in the *Review of Economic Studies* in 1945, accompanied by a commentary by David Champernowne (1945). From Champernowne's commentary, we learn that Sraffa had seen von Neumann's paper when Champernowne prepared his piece. However, until recently, we did not know whether Sraffa had already worked on problems such as joint production and the choice of technique—problems that figure prominently in von Neumann's contribution—prior to his acquaintance with the paper, and, if he had, what his results had been. We were thus also unable to say whether von Neumann's contribution had left any discernible traces in Sraffa's preparatory manuscripts, which were to grow into his 1960 book.

Since the opening of his unpublished papers and correspondence in the Wren Library at Trinity College, Cambridge, the situation has changed. Since, from an early stage, Sraffa tended to date his manuscripts, we know in most cases precisely when he tackled which question, formulated which hypothesis and arrived at which result. The available material sheds new light on the development of Sraffa's thoughts.

In this paper we make use of some of this material in order to contribute to a clarification of how Sraffa's reformulation of the 'classical' point of view in the theory of value and distribution relates to von Neumann's model. It should be stated right at the beginning that the available material is enormous and that we were able to review only a part of it. Therefore, it cannot be excluded that the collection contains other documents that are pertinent to the theme under consideration. These may provide additional support to the interpretation given, but they may also throw doubt on it. The reader should be aware of the preliminary character of this paper.

The composition of the paper is as follows. Section 2 provides a summary account of our view on the matter put forward in contributions published before we had access to the material (see Kurz & Salvadori, 1993; 1995, Chapter 13). In these publications we argued that, despite some obvious differences in the mathematico-analytical tools used by von Neumann and Sraffa, there are important conceptual equivalences in their approaches. It would, of course, be a pointless exercise to reiterate our earlier view were we of the opinion that, *vis-à-vis* Sraffa's unpublished manuscripts, this view can no longer be sustained. We will focus on the following issues: (i) the question of returns to scale; (ii) the asymmetrical treatment of the two distributive variables, the real wage rate and the rate of interest; (iii) fixed capital and depreciation; (iv) joint production; (v) the problem of the choice of technique, comparing what may be called the 'direct' and the 'indirect' approach; (vi) the difference between the rule of semi-positive prices (or the Rule of Free Goods), adopted by von Neumann, and the rule of strictly positive prices, adopted by Sraffa; and (vii) the treatment of

natural resources, especially land. In Section 3, we take a closer look at the gradual development of Sraffa's approach to the theory of value and distribution. We shall briefly summarize his investigation of 'systems of production' from the time in which he put down his first systems of equations in late 1927 to the publication of his 1960 book. It goes without saying that covering such a long period of time in a few pages necessitates a bird's eye view, focusing attention on a few aspects. Since one of the features of von Neumann's model is the multiple-products framework, we shall be especially concerned with when, and how, Sraffa himself dealt with the problem of joint production. Sections 2 and 3 set the stage for the rest of the argument. Section 4 is devoted to a brief discussion of Sraffa's role in Champernowne's attempt to come to grips with the economics of the von Neumann model in a comment that appeared in the *Review of Economic Studies*. Section 5 discusses some of the material contained in Sraffa's unpublished manuscripts and correspondence in which von Neumann is explicitly mentioned. Section 6 contains some concluding remarks.

2. Mathematical Differences and Conceptual Equivalences²

2.1. Returns to Scale

Von Neumann explicitly assumes constant returns to scale.³ Sraffa, on the other hand, stresses that in his analysis 'no such assumption is made', though he adds that 'If such a supposition is found helpful, there is no harm in the reader's adopting it as a temporary working hypothesis' (Sraffa, 1960, p. v). The different approaches to the question of returns follow largely from a difference in perspective: while von Neumann is concerned with a uniformly growing economic system and therefore needs this assumption, Sraffa's investigation 'is concerned exclusively with such properties of an economic system as do not depend on changes in the scale of production' (Sraffa, p. v). Hence, unlike von Neumann, Sraffa does not specify whether the surplus generated by an economy accumulates or is consumed (unproductively): there are no assumptions regarding saving and investment behaviour to be found in his book. Yet there appears to be nothing in Sraffa's approach which, as a matter of principle, would preclude the adoption of constant returns in combination with von Neumann's suppositions regarding saving and investment as a provisional working hypothesis, designed to shed some light on the economic system and its capacity to grow. (This does not mean that Sraffa would endorse such an extension of his equations.) The difference between the two is rather to be seen in the following: whereas von Neumann throughout his paper retains the simplifying assumptions

² The title of this section is a metamorphosis of the title of one of Schefold's (1980) papers. However, we do not enter into a discussion of the paper because Schefold does not deal with von Neumann's original article, but only with some of the literature triggered by it. Indeed, von Neumann's article is not cited in the paper. As regards the relationship between the literature under consideration and Sraffa's theory, Schefold sees conceptual differences and mathematical equivalences.

³ This may be considered the twin assumption to his setting aside scarce natural resources (see below).

just mentioned, Sraffa makes it clear, sometimes implicitly, that an analysis conducted in these is unnecessarily special and terms cannot cover empirically important cases.⁴

2.2. *Asymmetrical Roles of the Distributive Variables*

Von Neumann assumes that at the beginning of the (uniform) period of production, workers are paid a wage that covers no more than the ‘necessities of life’ (von Neumann, 1945, p. 2). Sraffa, at the very beginning of his book also adopts the assumption of a given subsistence wage, but later drops it. He takes into consideration that workers may receive a share of the surplus product (defined on the basis of a given subsistence wage) and then, after some deliberation, decides to treat wages henceforth as paid *post factum*, that is, at the end of the (uniform) period of production. This is tantamount to assuming wages to be paid entirely out of the surplus product. Sraffa is perfectly aware of the drawback of this approach, which risks losing sight of the indisputable subsistence aspect of wages and prevents one from considering the *real* wage rate as fixed. Hence, if the wage rate were still to be given from outside the system of production, it would have to be ‘in terms of a more or less abstract standard, and [would] not acquire a definite meaning until the prices of commodities are determined’ (Sraffa, 1960, p. 33), that is, until the system of equations is solved. Hence, Sraffa, unlike von Neumann, does not exclude the possibility of relative prices having an impact on the vector of commodities consumed by workers (and other income recipients). In these circumstances he decides to treat the rate of profits as the independent variable, because, ‘as a ratio, [it] has a significance which is independent of any prices, and can well be “given” before the prices are fixed’.

Thus, both analyses share a salient feature of the classical approach: they treat one of the distributive variables as exogenous and the other one (together with relative prices and, in the case of Sraffa, the rents of land) as endogenous. This *asymmetric* treatment of the distributive variables stands in striking contrast to the neoclassical theory of income distribution that attempts to explain wages, profits and rents simultaneously and *symmetrically* in terms of the supply of and the demand for the factors of production: labour, capital and land. This compels neoclassical authors to take the economy’s initial endowment of capital (and the other productive factors) as given. No such assumption is to be found in von Neumann or Sraffa. They do not explain distribution in terms of the relative scarcities of ‘capital’ and labour.

2.3. *Fixed Capital*

Both authors treat fixed capital within a joint production framework. This framework can be traced back to Robert Torrens and is also encountered in the

⁴ As Sraffa’s unpublished papers show, Sraffa had a foremost interest in elaborating a theory of accumulation, but first had to solve the problem of value and distribution. The latter turned out to be much more difficult than he expected when he began working on it in the late 1920s. As a matter of fact his constructive work was mainly absorbed by this problem. However, his manuscripts make very clear that, in conditions with ongoing technical progress, the depletion of stocks of exhaustible resources etc., there is no presumption that the economy will follow a steady-state path with the amounts of all capital goods used in the system expanding at a uniform rate of growth.

writings of David Ricardo, Thomas Robert Malthus and Karl Marx (see Sraffa, 1960, Appendix D). Von Neumann contents himself with the hint that the joint products method is capable of dealing with durable instruments of production: ‘wear and tear of capital goods are to be described by introducing different stages of wear as different goods, using a separate [price] for each of these’ (von Neumann, 1945, p. 2). Sraffa, on the other hand, devotes a whole chapter to the treatment of fixed capital employing this method (Sraffa, 1960, pp. 63–73). He demonstrates that this powerful method is not restricted to the ‘extremely simplified case’ of constant efficiency ‘but has general validity’ (Sraffa, 1960, p. 66); that the method allows one to ascertain the annual charge to be paid for interest and depreciation, and also to ascertain what the results derived imply for the theory of capital.

2.4. Joint Production

When we come to the two authors’ treatment of pure joint production we are confronted with two closely related issues that appear to indicate substantial differences between the two analyses: (i) while von Neumann adopts the Rule of Free Goods, in Sraffa’s book that rule is never mentioned; (ii) in contradistinction to von Neumann, Sraffa formulates his analysis of joint production in terms of equations rather than inequalities and assumes ‘that the number of independent processes in the system [is] equal to the number of commodities produced’ (Sraffa, 1960, p. 44). This assumption is rationalized in terms of the following argument, referring to a case in which two commodities are jointly produced by two different processes (or methods): ‘considering that the proportions in which the two commodities are produced by any one method will in general be different from those in which they are required for use, the existence of two methods of producing them in different proportions will be necessary for obtaining the required proportion of the two products through an appropriate combination of the two methods’ (Sraffa, 1960, p. 43, n. 2).⁵

⁵ It is interesting to notice that an argument in favour of the treatment of joint production in terms of ‘square’ systems of production, which is similar to Sraffa’s, had been put forward by F. Zeuthen in a critical discussion of the limitations and deficiencies of Gustav Cassel’s approach (see Zeuthen, 1933). With implicit reference to Chapter 16 of Book III of John Stuart Mill’s *Principles*, Zeuthen argues: ‘It is sometimes emphasized that here [i.e. in the case of joint production] the principle of cost is abrogated. This may be correct in the sense that the distribution of costs between products is not determined by the technical relations alone. ... However, on the assumed free mobility ... there will be a complete and automatic determination of prices. This can be imagined as follows. In the example of cattle-breeding there may exist two forms of business, one predominantly concerned with dairy products and requiring a lot of labour, the other predominantly concerned with the production of meat and thus requiring a larger live stock. ... [I]t follows that for each new method of production for a commodity there will be an additional magnitude as an unknown and a new cost equation which contributes to the solution of the system’ (Zeuthen, 1933, p. 15). And Sraffa, referring to the case in which two products are produced by means of a single method of production, maintains: ‘In these circumstances there will be room for a second, parallel process which will produce the two commodities by a different method. ... Such a parallel process will not only be possible – it will be necessary if the number of processes is to be brought to equality with the number of commodities so that the prices may be determined.’ And later he adds: ‘The same result as to the determination of prices which is obtained from the two commodities being jointly *produced* ... could be achieved

This elicits the following remarks. First, Sraffa, in accordance with the procedure adopted by the classical economists, in the case of single production and to some extent also in the case of joint production, approaches the theory of value and distribution in two steps. He analyses first the mathematical properties of a *given* system of production and only subsequently addresses the problem of which system will be chosen by cost-minimizing producers from a set of available alternatives. In carrying out the second stage of the analysis Sraffa compares alternative techniques one by one. This approach might be called 'indirect'. Von Neumann, on the other hand, is not concerned with investigating the mathematical properties of a given technique. He tackles at once the problem of the choice of technique from *all* the available alternatives. This approach might be called 'direct'. As is well known, in the case of single production (and in simple cases involving fixed capital), the two approaches produce exactly the same results (see, for example, Kurz & Salvadori, 1995, Chapters 5, 7).

Secondly, flukes apart, the particular assumptions that underlie von Neumann's model (all interest income is accumulated and the composition of workers' consumption does not depend on prices) generate a situation in which Sraffa's premise holds—in the sense that the number of processes is equal to the number of commodities with a positive price (see Steedman, 1976; Schefold, 1978, 1980; Bidard, 1986). However, with less special assumptions it cannot be presumed that the number of independent processes in the system is always equal to the number of commodities produced. Sraffa's justification of this premise in terms of the 'requirements for use' is valid only in some circumstances.⁶

2.5. *The Rule of Free Goods*

One can distinguish between the application of the Rule of Free Goods (or the assumption of 'free disposal') to 'original' factors of production, in particular different qualities of land, and to produced commodities. Here we shall deal only with the latter case, whereas the former will be touched upon below in the subsection on 'Land'.

Footnote continued—

if the two commodities were jointly produced by only *one* process, provided that they were *used* as means of production to produce a third commodity by two distinct processes; and, more generally, provided that the number of independent processes in the system was equal to the number of commodities produced' (Sraffa, 1960, pp. 43–44; Sraffa's emphases). We have no evidence that Sraffa was familiar with Zeuthen's work.

⁶ The fact that these aspects of Sraffa's analysis cannot be sustained must not, however, be taken, wrongly, to imply the irrelevance of his approach to joint production. The indirect approach can still be useful when a square cost-minimizing technique obtains, which is necessarily the case in some significant circumstances, but not always (see, for instance, Kurz & Salvadori, 1995, pp. 236–240, and the whole of Chapter 9 on jointly utilized machines). Moreover, with universal joint production the indirect approach can be elaborated in such a way that it replicates the results obtained with the direct approach, although in terms of analytical convenience it is inferior to the latter (see Salvadori, 1985).

Sraffa points out that while, with single production, no price can become negative as a result of the variation of the wage rate between zero and its maximum value, given the system of production, ‘it may be said at once, however, that this proposition is not capable of extension to the case of joint-products. ... The price of one of them might become negative’ (Sraffa, 1960, p. 59). Sraffa comments on this possibility as follows:

This conclusion is not in itself very startling. All that it implies is that, although in actual fact all prices were positive, a change in the wage might create a situation the logic of which required some of the prices to turn negative: and this being unacceptable, those among the methods of production that gave rise to such a result would be discarded to make room for others which in the new situation were consistent with positive prices. (Sraffa, 1960, p. 59)

This passage witnesses that Sraffa was clear about the fact that the positivity of prices cannot be guaranteed if there is no choice of technique. As to the substance of Sraffa’s suggested way out of the impasse arising from the negativity of the price of a joint product, it can be argued that it is tantamount to the *ad hoc* assumption that there is always at least one process of production which, if adopted, makes the phenomenon of negative price disappear. This assumption, as peculiar as it may seem at first sight, is however no less *ad hoc* than the assumption of free disposal. In fact, the latter is equivalent to the assumption that, for each process producing a given product, there is another process that is exactly identical to the first one except that the product under consideration is *not* produced (see Kurz & Salvadori, 1995, Section 5 of Chapter 7, where *costly disposal* is also introduced along the same lines, and Section 2 of Chapter 8).

2.6. Land and Labour

While the two authors seem to disagree with regard to whether or not the Rule of Free Goods is applicable to produced commodities, they appear to agree with regard to land. Von Neumann assumes that ‘the natural factors of production, including labour, can be expanded in unlimited quantities’ (von Neumann, 1945, p. 2).⁷ Yet, this does not make him treat all these factors alike. Rather, he applies the Rule of Free Goods in the same way as the classical economists. He singles out labour as the only factor that is exempt from that rule; all other primary factors, although needed in production, ‘disappear’ from the scene because they are taken to be non-scarce: they fetch a zero price. Labour is assumed to receive

⁷ Assuming that natural resources are non-scarce is, of course, not the same thing as assuming that there are no natural resources at all. Von Neumann’s model is frequently misinterpreted in the latter sense. However, with the system growing forever, the point will surely come where some natural resource(s) will become scarce. Surprisingly, von Neumann simply ignores this. As Professor Samuelson has pointed out to us in private correspondence, ‘More by inadvertance than conscious intention, v.N. failed to emphasize the basic classical notion of land resources as unproducible or diminishable.’ The total neglect of the problem of scarce primary resources such as land distinguishes his analysis in fact from the approaches of both the classical and the neoclassical economists. For a possible explanation of this neglect, see Kurz & Salvadori (1995, Chapter 13, Section 7).

an exogenously given wage bundle that is independent of the degree of employment.⁸ Sraffa devotes a whole chapter to ‘Natural resources which are used in production, such as land and mineral deposits’ (Sraffa, 1960, p. 74), and makes it clear that as long as they are available in abundance they will not yield a rent to their owners. It is only when they are scarce that they assume economic weight. The scarcity of a resource, Sraffa points out, is generally reflected in the coexistence of more than one method utilizing it or more than one method using the product produced by means of it. Sraffa’s concern, it should be stressed, is exclusively with land, which is treated as a renewable resource whose quality is taken not to change irrespective of the way it is used, whereas exhaustible resources and general renewable resources are implicitly set aside.

3. The Development of Sraffa’s Analysis

In the preface of his 1960 book, Sraffa points out: ‘Whilst the central propositions had taken shape in the late 1920’s, particular points, such as the Standard commodity, joint products and fixed capital, were worked out in the ‘thirties and early ‘forties. In the period since 1955, while these pages were being put together out of a mass of old notes, little was added, apart from filling gaps which had become apparent in the process’ (Sraffa, 1960, p. vi). This is confirmed by Sraffa’s unpublished manuscripts. In what follows we shall provide a brief account of the development of his thoughts over time, paying special attention to those aspects that are pertinent to the theme of this paper.

At the beginning of his academic career, economics to Sraffa was essentially Marshallian economics. He was critical of it, but originally appears to have been of the opinion that it was worth attempting to shed its weaknesses and develop its strengths. He despised especially the subjectivist elements of Marshall’s theory of value and contemplated the possibility of purging the analysis of them (see D3/12/7:114).⁹ His starting point was not, as some commentators have speculated, Marx and the ‘transformation problem’. He objected against the labour theory of value that it involved a ‘corruption’ of the theory of value based on the concept of ‘physical real cost’, which he considered to provide an appropriate starting point for the theory of value and distribution (cf. D3/4: 2; see also D3/11: 79–80). In another note he emphasized that there is no ‘objective difference’ between the labour of a wage earner and of a slave, of a slave and of a horse, of a horse and of a machine, and added: ‘It is a purely mystical conception that attributes to labour a special gift of determining value’ (D3/9:89).

It was only after he had developed his first systems of equations that Sraffa saw that in special cases the labour theory of value gave essentially the same answers as his own conceptualization. This finding appears to have prompted

⁸ ‘At most, one could say that a “Rule of Zero ‘Excess’ Wages” is applied because labour is less than fully employed’ (Steedman, 1987, p. 419).

⁹ References to Sraffa’s unpublished papers and correspondence follow the catalogue prepared by Jonathan Smith, archivist. We should like to thank Pierangelo Garegnani, literary executor of Sraffa’s papers and correspondence, for granting us permission to quote from them.

him to study more carefully the classical economists and Marx.¹⁰ His interest in Marx as an economic theorist thus appears to have been a consequence of, rather than a precondition to, his own thoughts on the matter. The evidence suggests that it was only after the development of his first systems of equations in the second half of 1927 that Sraffa started systematically to study Marx's contributions to political economy. It was not until the early 1940s that he came across Ladislaus von Bortkiewicz's criticism and 'rectification' of Marx's argument concerning the so-called 'transformation' of values into prices of production in the third volume of *Capital*. He excerpted Bortkiewicz's papers with great care and put down numerous critical remarks. By that time, Sraffa had already gone a long way in developing his own point of view.

3.1. *Production as a Circular Flow*

Here, we cannot enter into a detailed discussion of the development of Sraffa's views, which changed considerably over time, especially after he had begun to grasp the analytical structure of the classical theory of value and distribution. As a consequence, his understanding of the marginalist theory, and its deficiencies, also underwent a change. While Sraffa retained his critical attitude towards the subjectivist part of that theory, the main target of his criticism now became the explanation of profits in terms of the supply of, and the demand for, a factor called 'capital'. It was in the late 1920s that Sraffa, all of a sudden, must have seen a glimpse of the alternative point of view that fundamentally changed his outlook—a change that is also reflected in his 'Lectures on Advanced Theory of Value' (D2/4). In one place, Sraffa notes that contrary to his earlier opinion, even with constant returns to scale, value in Marshall's theory cannot be assumed as given and constant, because it does not depend only on real physical costs, but also on the distribution of income between wages and profits. His equations indicated that a change in that distribution will generally change relative values.

He appears to have developed his systems of equations from scratch. From the beginning he assumed that commodities are produced by means of commodities, that is, he conceptualized production as a circular flow and not, as the Austrians had done, as a one-way avenue leading from original factors of production to final goods. For example, at the end of November 1927 he put down equations representing two industries without and with a surplus (see D3/2: 32–35). In the case where there is no surplus, exchange ratios between commodities are fully determined by the physical scheme of production and

¹⁰ In February 1930, the Royal Economic Society assigned Sraffa to the task of editing David Ricardo's works and correspondence. As we know, Sraffa immediately took up the work and put a great deal of effort into it. However, for a while he appears to have been of the opinion that he could also carry on with his constructive work, albeit at a much reduced speed. Yet soon Sraffa got overwhelmed by the new task, which absorbed all his energy and forced him to interrupt his constructive work.

reflect physical real costs. When there is a surplus, things are more complicated. One of the systems with a surplus Sraffa discussed is given by

$$\begin{aligned} 11A &= 3A + 9B \\ 13B &= 7A + 3B \\ S &= 1A + 1B \end{aligned}$$

where A and B indicate the prices of two commodities and S the value of the surplus product of the system as a whole. Sraffa observed that these equations are ‘contradictory’ (Sraffa, 1960); in another document he added that ‘the problem is overdetermined’ (D/3/11: 17). In the case with a surplus, a rule is needed according to which the surplus is distributed. It is only after this rule has been fixed that relative prices can be ascertained. In conditions of free competition and setting aside the problem of scarce natural resources, such as land, the surplus is distributed according to a uniform rate of return on the capital advanced in each sector of production.

As has already been stressed, in Sraffa’s argument, labour values at first played no role whatsoever. There was indeed no analytical scope for them, because, as Sraffa demonstrated, the problem of value and distribution is fully settled in terms of the two sorts of data contemplated: (i) the system of production (and productive consumption) in use; and (ii) the rule governing the distribution of income. The argument could be elaborated without ever referring to labour values. However, Sraffa saw that, in exceedingly special circumstances, that is, essentially those that had already been pointed out by Ricardo and Marx, the exchange ratios are proportional to the relative quantities of labour embodied in the different commodities. The special circumstances are: first, the case in which the rate of profits is equal to zero, and, second, the case in which the proportions of direct labour to labour embodied in the means of production are identical in all industries. In general, the exchange ratios differ from the ratios of labour embodied in the different commodities. Sraffa therefore suggested that the special constellation in which profits vanish might be considered from a different perspective and spoke of the ‘Value Theory of Labour’ rather than the ‘Labour Theory of Value’.

An early concern of Sraffa’s was the determination of what he later called the maximum rate of profits of a given system of production; that is, that rate which is compatible with some minimum (subsistence) real wage rate. Next, he began to study systematically the problem that had bothered Ricardo until the end of his life: the impact of a rise (or fall) of the real wage on the rate of profits and relative prices, given the system of production. That problem turned out to be much more intricate than economists had generally realized. Sraffa stressed: ‘In such a world, where everything moves in every direction ... one sympathizes with Ricardo in his search for an “invariable measure of value”. In a universe where everything moves we need a rock to which to cling, a horizon to reassure us when we see a brick falling that it is not we who are going up — nor that we are falling when we see a balloon rising’ (D3/12/52: 17).

To facilitate the study of changes in prices as distribution changes, Sraffa, in a series of steps, groped his way to the concept of the ‘Standard commodity’, which proved to be a powerful tool of analysis. As Sraffa stressed, while this

particular standard of value ‘cannot alter the system’s mathematical properties’, it is explicitly designed to ‘give transparency to a system and render visible what was hidden’ (Sraffa, 1960, p. 23). The first important mathematical property of a given system is its maximum rate of profits. The Standard system allows one to ascertain that rate in a straightforward manner. It also provides ‘tangible evidence of the rate of profits as a non-price phenomenon’ (D3/12/43: 4), an observation which echoes Ricardo’s contention that ‘the great questions of Rent, Wages and Profits ... are not essentially connected with the doctrine of value’ (Ricardo, 1951–73, *Works*, Vol. VIII, p. 194). The Standard commodity is essentially a tool of analysis that allowed Sraffa to see through the intricacies of the movements of relative prices as income distribution changes, given the technique in use. Sraffa could have obtained the same results by using the Perron–Frobenius theorem; in fact, Sraffa’s demonstration of the existence and uniqueness of the Standard commodity can be considered a (not fully complete) proof of this theorem (see Kurz & Salvadori, 1993).

3.2. Joint Production

Sraffa had already started to tackle the problem of joint production at an early stage of his work. This is not surprising, given his concern with the difficulty fixed capital introduces into the theory of value: while the circulating part of the capital advances contributes entirely to the annual output, the contribution of the durable part is less obvious and can only be imputed in correspondence with what may be considered the wear and tear of fixed capital items. Sraffa sought to solve the intricate problem by reducing fixed capital to circulating, which implied that each vintage of a particular type of durable capital good had to be treated as a separate commodity. The suggested reduction involved the adoption of a general joint products framework.

In November 1927, Sraffa considered the case of the overproduction of one of the joint products and put forward a clear formulation of the Rule of Free Goods: ‘Joint products: they are *always* assumed to be slightly variable, and therefore to have a marg. cost (both cover the whole: Wicksteed, or Euler) [...] — Well, as we are in const. returns, that is the cost of each — If absolutely invariable, probably *only* one would have a price: the one which is not wanted (at whatever price) in that amount, would be gratis’ (D3/12/11: 25). However, later he appears to have abandoned that rule. At any rate, he did not adopt it in his book. His preparatory manuscripts document that he contemplated other options. In a note dated 27 October 1943 he discussed the case of ‘Joint Products (when only *one* equation exists)’. The reference is to a process that produces jointly two products. Sraffa points out that the conventional approach is to take the aggregate cost as given. He objects to this assumption on the grounds that if ‘one of the products is itself part of the cost ... the aggregate cost cannot be known in the first instance.’ He adds:

When this difficulty does not arise, the margin[all]ist has two alternative methods at his disposal: 1) Marginal products, when the proportions of production are variable — 2) Marginal utilities, when the proportions are fixed. — The first is out of production, the second out of consumption. *Similarly* with our approach.

He substantiates the latter remark in terms of the following two possibilities. First, there are two joint production methods producing the two products, say A and B, in different proportions. Secondly, there is only one joint production method, but one of the products is used as a means of production in producing a third product, say C, which is generated by means of a single products process. Now, if there is a second method to produce the third product, but using a different amount of the input per unit of output, we may again, Sraffa contends, get a system in which the problem of overproduction vanishes (D3/12/35: 41).¹¹

It deserves also to be mentioned that, as early as around the turn of 1942–43, Sraffa discovered the possibility of negative costs or values in joint production systems (cf. D3/12/28). In addition, in February 1946 he stated that in such systems, ‘when we change r [the rate of profits] from its actual value, and make it, say, $= 0$, we may obtain negative Values’ (D3/12/16: 35).

We may summarize our findings as follows. By the time of the publication of the English translation of von Neumann’s paper, Sraffa had already elaborated important elements of his analysis. These concerned, first and foremost, the case of single production, that is circulating capital only, but it was by no means restricted to it. He had already worked for a considerable time on various aspects of joint production and fixed capital and had come up with some remarkable results.

4. Champernowne’s Commentary

Kaldor, as mentioned above, stimulated the publication of an English version of von Neumann’s paper and was also concerned with rendering the mathematically demanding piece attractive to an audience of economists. A first step in the pursuit of this goal appears to have been the adaptation of the paper’s title (cf. Kaldor, 1989, p. x), which, in a literal translation of the original German, would have been ‘On an Economic System of Equations and a Generalization of Brouwer’s Fixed Point Theorem’. The second part of the title, which reflects von Neumann’s assessment that the main achievement of the paper consisted of the generalization of a mathematical theorem, was dropped entirely, and the neutral term ‘economic system of equations’ was replaced by the not-so-neutral term ‘model of general economic equilibrium’.

The second step consisted of asking David Champernowne, ‘the most mathematically-minded economist I knew, to write an explanatory paper *ad usum delphini*, for the use of the semi-numerates, to appear alongside it in the *Review of Economic Studies*’ (Kaldor, 1989, p. x).¹² In a footnote to the introduction of his paper Champernowne thanks several people. Interestingly, the footnote in the galley proofs of Champernowne’s paper in Sraffa’s library is different from the published one. The former reads:

This note is the outcome of conversations with Mr. N. Kalder [sic] and Mr. P. Sraffa, to whom many of the ideas in it are due. I am also indebted to Mr.

¹¹ Sraffa’s contention that, in this case, all prices are strictly positive cannot be sustained in general.

¹² It is interesting to note that in the title of Champernowne’s (1945) paper the title of the English version of von Neumann’s paper is referred to incompletely: the adjective ‘general’ is left out.

Crum of New College, Oxford, for his helpful comments on the mathematics in Dr. Neumann's article. (Sraffa's library, item 4674)

The published version reads as follows:

This note is the outcome of conversations with Mr. N. Kaldor, to whom many of the ideas in it are due. I am also indebted to Mr. P. Sraffa of Cambridge and to Mr. Crum of New College, Oxford, for instruction in subjects discussed in this article. (Champernowne, 1945, p. 10, n. 1)

Whereas according to the early version of the footnote Kaldor and Sraffa were to be credited with the ideas in the commentary, now it is only Kaldor. In a letter to Sraffa dated 1 April 1947, accompanied by an offprint of his paper,¹³ Champernowne sets the record straight:

I didn't like to put more than that about you in the footnote, but of course you told me all about (a) cost-theory of value (b) the A.G.D. Watson price-matrix theory: even if my note didn't exactly express your ideas. I think that Neumann's article solves the problem.

We have been unable to pin down what Champernowne meant when referring to the 'A. G. D. Watson price-matrix theory'. Be that as it may, it should come as no surprise that, in his interpretation, von Neumann's model emerges as one characterized essentially by 'classical' features.

In the course of his investigation Champernowne puts forward several concepts and raises a number of issues that we re-encounter in Sraffa (1960). Thus, Champernowne uses the notion of 'system of production' (Champernowne, 1945, p. 14), which figures prominently in Sraffa's analysis. He notes that, in the von Neumann model, the role of the 'worker-consumer' can be compared with that of a 'farm animal', e.g. a work horse, whose costs consist of his 'fodder, stabling, etc.' (Champernowne, 1945, p. 12), an analogy that recurs in Sraffa's formulation in Chapter II of his book: 'We have up to this point regarded wages as consisting of the necessary subsistence of the workers and thus entering the system on the same footing as the fuel for the engines or the feed for the cattle' (Sraffa, 1960, p. 9). The rate of interest, Champernowne stresses, 'depends on the technical processes of production which are available' (Champernowne, 1945, p. 12); Sraffa, on the other hand, elaborates the Standard system with R as the 'Standard ratio or Maximum rate of profits' representing a ratio between quantities of commodities (Sraffa, 1960, p. 22). Champernowne raises the question of what would happen if the real wage were higher than originally assumed and concludes that 'there will be a change in the equilibrium conditions ... with a lower rate of interest and a lower rate of expansion' (Champernowne, 1945, p. 16); this foreshadows the inverse relationship between the rate of profits and the real wage rate analysed by Sraffa.

In the above-mentioned copy of the galley proofs of Champernowne's paper in Sraffa's library, there are annotations in Sraffa's hand. It is perhaps interesting to note some of the passages marked. These are:

- (i) 'no saving was carried out by the workers whereas the propertied class saved the whole of their income' (p. 12; this is indicated as one of the

¹³ See item 4676 of Sraffa's library; Champernowne's letter is inside the pages of the offprint.

- ‘several drastic simplifying assumptions’ introduced in order ‘to make it possible for quasi-stationary state equilibrium to exist in the model’);
- (ii) ‘Since Dr. Neumann’s results only relate to a quasi-stationary state, the utmost caution is needed in drawing from them any conclusions about the determination of prices, production or the rate of interest in the real world’ (p. 15; in the published version ‘Dr. Neumann’ is substituted by ‘v. Neumann’);
 - (iii) with a higher real wage rate ‘there will be a change in the equilibrium conditions, and the position of quasi-stationary equilibrium will change to one with a lower rate of interest and a lower rate of expansion’ (p. 16);
 - (iv) ‘The rate of interest will be determined as the greatest rate of expansion possible if all income from property is saved ... [even if part of the income from property were spent on consumption, and not saved, the rate of interest would not necessarily be much affected] it might still be *approximately* equal to the greatest expansion rate that *would* have been possible *if* all income from property had been saved’ (p. 16);
 - (v) ‘here, perhaps for the first time, is a self-contained theory of the determination of prices, ignoring the second approximation’ (p. 17; the ‘second approximation’ refers to the introduction of ““special cases” such as “the possibility of increasing returns” and “consumers’” choice as an independent factor in the direction of productive activity’, which ‘in traditional economics’ are ‘at the centre of the theory’);
 - (vi) ‘It is expressly assumed that every good is involved (either as input or as output) in every process’ (p. 18; Champernowne is critical of this assumption);
 - (vii) ‘It should be noted that although in the model the equilibrium rate of interest is uniquely determined, the system of prices and outputs are not *uniquely* determined: there may be any number of possible equilibrium positions. But each must satisfy the rules set out in section 2 above’ (p. 18).

As regards the bold premises (i), (ii) and (vi) that underlie von Neumann’s model, Sraffa can safely be assumed to have shared or even inspired Champernowne’s critical attitude. Most interesting is perhaps (v). We do not know whether it was due to Sraffa’s ‘instruction’ that Champernowne in his commentary put forward the idea of a ‘first’ and a ‘second approximation’ in the theory of prices. Accordingly, factors such as a shift in demand ‘may conveniently be considered as the “special cases” of price-theory, to be introduced in the *second approximation*; and not, as is common in traditional economics, at the centre of the theory. For the basic influences determining equilibrium prices v. Neumann’s model provides a novel approach; here, perhaps for the first time, is a self-contained theory of the determination of prices, ignoring the second approximation’ (Champernowne, 1945, p. 17; emphasis in the original). Champernowne is of the opinion that von Neumann’s ‘first approximation’ is particularly powerful with regard to intermediary goods. In a footnote he adds: ‘And even in the case of final consumers’ goods, the prices ... are *largely* to be explained by the technical conditions of production, rather than “marginal utility”’; then follows, in

brackets, the adjunct: ‘The exceptions being joint products, or commodities with largely increasing or decreasing cost’ (Champernowne, 1945, p. 17, fn.).

In addition, with respect to the other items, we cannot say whether or not they have been prompted by Sraffa’s ‘instruction’. However, it should be noted that items (iii) and (iv) concern the fact that the relationship between the wage rate and the rate of interest is decreasing; and that—with constant returns to scale and joint production, as assumed by von Neumann—this relationship is not much affected by the quantities produced. There is evidence that Sraffa was well aware of these facts in the 1940s with respect to single production. (He was perhaps inclined to think that they carry over to systems with joint production, but this needs to be checked.)

Item (vii) is more problematic than the others. In Sraffa’s book, a whole chapter is devoted to the ‘Uniqueness of the Standard System’ in single production (Chapter V), but there is no attempt to prove the uniqueness of the prices even in the cases in which this proof would be possible (the non-substitution theorem).

5. Sraffa on von Neumann

In the preface to *Production of Commodities*, referring to ‘the disproportionate length of time over which so short a work has been in preparation’, Sraffa remarks: ‘As was only natural during such a long period, others have from time to time independently taken up points of view which are similar to one or other of those adopted in this paper and have developed them further or in different directions from those pursued here’ (Sraffa, 1960, p. vi). One of the authors Sraffa may have had in mind when writing these lines was John von Neumann. The question then is, to what extent did Sraffa absorb or reject the ideas put forward by von Neumann?

In this section we take account of statements in Sraffa’s papers in which he explicitly mentions von Neumann. Note, however, that it is not claimed that the following discussion exhausts the issue. We shall begin with a discussion of the assistance Sraffa received from Alister Watson, one of Sraffa’s ‘mathematical friends’ (see D3/12/46: 49) whom he thanked in the preface of his book (Sraffa, 1960, p. vii). Watson played a significant part in two periods of Sraffa’s work on his book: first in the 1940s; and then in the period since 1955, when Sraffa put together the text out of a mass of old notes, including the proof-reading stage.¹⁴

5.1. Alister Watson’s Visits

In January 1947, Alister Watson visited Sraffa in Cambridge; Sraffa took notes of their discussion (D3/12/44: 4, 6). The main question under consideration was the uniqueness of the maximum rate of profits R :

¹⁴ For a detailed discussion of the collaboration of Frank Ramsey and Alister Watson with Piero Sraffa, see Kurz & Salvadori (2000).

I. Q-system: several all-positive solutions. The only solution I have considered gives a value 0 to the qs of all *non-basic* processes.

However, suppose that one (or more) of the non-basic commodities [wheat] uses itself in its own production in a proportion greater than that of the basics taken as a whole (in other words, its own internal R is *smaller* than the R of the basic group), then there is another solution: for this non-basic commodity uses in its own production some basic ones, thus diminishing the ratio of basic means of production to basic products.

If, on the other hand, the internal R of the non-basic is larger than the R of the basic group, there is only one all-positive solution, with the q 's for non-basics = 0.

N.B. This has its symmetrical case in the P-system. If some of the non-basics have an own internal R *larger* than the basics group, there are alternative solutions with all the basic p 's = 0, and bigger values of R. (D3/12/44: 4)

The notes continue on page 6, whereas page 5 includes a note added by Sraffa on 23 February 1955. Let us first report the end of the note of 1947.

We can thus sum up:

There are several non-negative systems of roots of the Q-system. The system with the largest value for R has all zero values for the qs of the non-basic processes.

There are several non-negative systems of roots of the P-system. All these systems, except the one with the smallest value for R, have all zero values for the ps of the basic commodities.

[N.B. (1) The largest R of the Q-system is equal to the smallest R of the P-system. — (2) The proposition referring to the Q-system assumed that non basics have a *smaller* internal own ratio than r-basic; that referring to P system assumes it *larger*] (D3/12/44: 6)

In the note added in 1955, Sraffa mentioned von Neumann:

We can avoid all these complications by, from the start, removing 'manually' all the non-basic equations and dealing with a system composed exclusively of basic commodities [these to be defined, before the removal, as comm.s which directly or indirectly enter all the others]* and then we can say that there is only one all-positive [and not merely non-negative) solution for the ps and the qs .

[N.B. One point which needs clearing about the Watson alternative solution is this: does it remain true that if we multiply the equations by *any* pair of solutions of the ps and qs , which is not the all-positive pair of solutions, the sum of all the equations is *null*?]

* For practical application this good enough. But discuss in a note the abstract possibilities of this not being so, e.g. of the system falling into two or more self-contained (self-basic) groups of commodities — as if one lumped together the equations of two countries which have no commercial relations (& treating, of course, iron in country A as a different commodity from iron in B).

The more 'elegant' system of solving the complete system (with qs of non-basics = 0) can be discussed here with the Watson difficulties (query, did he derive them from von Neumann?) (D3/12/44: 5)

In April 1955, Watson visited Sraffa again, but in the note concerning that visit (D3/12/58: 8–9) there is no mention of von Neumann. In February, Sraffa was no doubt annotating his previous notes in preparation for the visit of Watson in

April. However, we do not know whether Sraffa asked Watson about von Neumann and got an irrelevant answer or whether he himself thought that the question was actually not interesting.

5.2. *The Correspondence between Hicks and Sraffa*

In a letter dated 3 September 1960 (D3/12/111: 267–268), John Richard Hicks commented on *Production of Commodities*, which he had just finished reading. In his comment he referred to von Neumann and pointed out several similarities between the constructions by von Neumann and Sraffa. We quote the letter in full:

My dear Piero,

When I got back in mid-July from four months in the Orient, I found your book waiting for me; I did not immediately write to thank you, as I wanted to read it first, and it has taken quite a number of weeks clearing off various kinds of back-logs before I could get down to the attempt to absorb anything new. It is only now that I have been able to read it — not yet in detail but sufficiently to have a general impression and to have something that I very much want to say.

You tell us that your work on the subject goes back a long way — you mention Frank Ramsey; is it possible that it was somehow through you and your mathematical friends that von Neumann got onto what is in so many ways a similar construction (it is understood that his paper was originally given at Princeton in 1932)? I have never been able to understand how he should have hit on it out of the blue. Formally, I believe, your standard system is identical with the von Neumann equilibrium, though it arises in response to a different question. But the model, even to the treatment of fixed capital, is exactly the same.

I am myself intensely interested in the pulling-apart, which you have performed, of the system without and with joint production. I have lately run into this matter myself in two different contexts.

One was over the paper on ‘Linear Theory’ which is to appear as a survey in the December EJ. In the first draft of my paper I followed Dorfman, Samuelson and Solow (Linear Programming chs. 9–10), in a statement of the Samuelson ‘substitution theorem’ to the effect that (under constant returns to scale), with labour as the only ‘outside’ — I think you would say ‘non-basic’ — input, technical coefficients are determined independently of demands, so that the system operates under constant costs. When I sent this in to the editors, Robin Matthews pointed out to me that I had not allowed for joint production. (Had he read your MS?) Then, on my travels, I got to California; there I was told by Arrow that he and Koopmans had noticed the gap in the Samuelson argument, though the mathematics in which they had wrapped up their point (chs. 8–9 of the ‘Activity Analysis’ book) was too ‘opaque’ for me to be able to understand it. I have now found a bit of geometry which makes the whole thing fairly clear, and have put it into my revised version.

The other, even more directly relevant, concerns the von Neumann growth model itself and the ‘Turnpike Theorem’ that Samuelson and Solow have based upon it. Here again I started from the treatment in the Do.S.So. book (chs. 11 and 12), which in this case I did not find at all convincing. In an endeavour to puzzle the thing out for myself, I made precisely the sim-

plification which you make in your Part I (this was of course not made by von Neumann and the others, but what was true in the general case ought to be true in the special case also, so one could use it, as you use it, as a means of finding one's feet). But when I did so, I got into trouble, and so started quite a controversy (so far consisting only of letters and circulated papers but soon to get into print). I thought at one time that I had found an exception to the Samuelson theorem, and when I was in California last March I still had a paradox for the pundits, to which the answer was not found obvious. In the end, while working with Morishima in Japan, I got it out. What had not been noticed is that under the assumption of no joint production (but not when there is joint production) there is a tendency to constant costs in the von Neumann system (effectively your system with no non-basic elements). This is evidently related to the Samuelson substitution theorem, though it is not the same thing. This Turnpike stuff will be appearing in the February Review of Economic Studies, together with related contributions from Samuelson, Morishima and probably others. Since this has not yet gone to press (unlike the EJ survey, which is out of my hands) I shall certainly add a reference to your work, which is so clearly to the point. But there is certainly much to be done in fitting together your approach with those of others. It will no doubt take much time before that is properly done; it is however quite an exciting job to have before us.

Economic theory (teachable economic theory, at least) was getting just a bit boring lately; for the second time in your life you have livened it up again. Thank you.

Yours ever,
John Hicks.

The draft of Sraffa's reply is dated 8.9.60 (D3/12/111: 269). We have yet to check whether the letter was sent and received by Hicks. Here is the text of the draft.

My dear John,

I was delighted to receive your letter as there was no one whose reaction I was more interested to have.

I have not here the books you refer to on particular points, and I expect to be writing to you again when I get back to Cambridge.

The reason for the analogy between the several constructions seems to me to lie in their having a common source, although by devious ways, in the old classical economists (before their successors introduced the 'cost of production' theories, e.g. Mill, Cairnes etc.: that is undoubtedly the case for the treatment of fixed capital as a joint product. There are however important differences with the von Neumann construction, and the saddle point and the 'free goods' are peculiar to it: I never succeeded in getting quite clear on these points of his but although I am not certain I believe they are related to his treatment of what I call 'non-basic' goods.

The answer to your query concerning the editors of the E.J. is that Austin Robinson read the MS as a Syndic of the Press, but Robin Matthews did not. I really am writing only to say how grateful I am for your having taken so much trouble about my little book and if I can contribute anything on the other matters you refer to, I shall write again.

5.3. Comments

Sraffa was certainly not able to read the mathematics of John von Neumann, but he was able to recognize that von Neumann shared essentially his own outlook. In both models commodities are produced by means of commodities. Profits (interest) are taken to ‘come out of the surplus produce’, to use Ricardo’s expression (*Works*, Vol. II, pp. 130–131): given the real wage rate, profits are a residual left after the wage goods in the support of labourers and what is necessary for the replacement of the used up means of production have been deducted from the annual output. In both models, the rate of profits is not a scarcity index of a factor called ‘capital’.

This interpretation is similar to that given by Joan Robinson in a letter to Peter Newman. This letter is dated 29 May 1962; a copy is in the Sraffa papers (D3/12/111: 304).

Your detective work on the influence of von Neumann in Cambridge seems a bit illogical. The reason why Sraffa could explain him to Champernowne was that Sraffa had already made the discovery. Not that I want to be ungrateful to von Neumann. His model is beautiful and it is very useful for dealing with those people who cannot see a simple point unless it is put in a complicated way.

6. Conclusions

This paper argues that Sraffa’s 1960 book and von Neumann’s 1937 paper share essentially the same outlook and exhibit remarkable conceptual parallels. Both contributions belong to the ‘classical’ approach to the theory of value and distribution, characterized by an asymmetric treatment of the distributive variables, the rate of return on capital and the real wage rate. In addition, the paper presents and discusses some material from Sraffa’s hitherto unpublished papers and correspondence which is pertinent to the problem under consideration. From Sraffa’s unpublished papers it can be seen that he was not able to understand the mathematics of von Neumann, but also that he understood perfectly well that they started essentially from the same theoretical point of view, that is, the one of the classical economists. This is also why Sraffa was able to discern in von Neumann several aspects which he, Sraffa, had already, at least in part, elaborated himself. When Sraffa came across the von Neumann model his own analysis was already quite advanced.

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