Marx's transformation problem and Pasinetti's vertically integrated subsystems

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Pasinetti (1988) constructs a 'complete generalization of Marx's "transformation problem"', in the context of a non-uniform growth model, by proving that production prices are not proportional to the physical quantities of labour supplied to 'vertically hyper-integrated subsystems'. Pasinetti therefore restricts the labour theory of value to a pre-institutional stage of analysis. In this paper I demonstrate that the transformation problem dissolves once we consider the vertically integrated subsystems induced by the specific institutional setup of a capitalist economy. I prove that production prices, both in steady-state and non-uniform-growth models, are proportional to the physical quantities of labour supplied to 'vertically super-integrated subsystems'. In consequence the labour theory of value, suitably generalized, also applies at the institutional stage of analysis.

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Piero Sraffa (1960, ch. 3) demonstrated that the natural prices of reproducible commodities necessarily vary with the distribution of income due to the 'inequality of the proportions in which labour and means of production are employed in the various industries', whereas the real costs of production of commodities, measured in terms of labour, do not. In consequence, labour costs cannot fully explain the structure of natural prices. This explanatory gap creates various problems for the classical labour theory of value, most notably Karl Marx's transformation problem (e.g. see Seton, 1957; Desai, 1988; and Hunt and Glick, 1990). Many authors interpret Sraffa's analysis to imply that the labour theory of value is, at best, incomplete, or worse, logically incoherent (e.g. Samuelson, 1971; Lippi, 1979; Steedman, 1981).

Luigi Pasinetti, a follower of Sraffa, offers a different interpretation. He proposes a separation thesis[1] (Pasinetti, 2007, ch. 9) that orders the study of economic systems into a pre-institutional or '"natural" stage of investigation', concerned with 'the

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[1] Pasinetti calls it a theorem, but since no proof is involved, I prefer to call it a thesis.

foundational bases of economic relations' that reveal the fundamental constraints that any economic system must satisfy, followed by an 'institutional stage' (Pasinetti, 2007, p. 276), which is 'carried out at the level of the actual economic institutions' (Pasinetti, 2007, p. 275), which identifies how the constraints manifest in specific institutional setups. Pasinetti's attitude to the labour theory of value is shaped by this separation.

Pasinetti argues, in a series of works (e.g. Pasinetti, 1981, 1988, 1993), that the labour theory of value, rather than being incomplete or incoherent, is a powerful analytical tool at the pre-institutional stage of investigation, and therefore 'has to be taken as providing a logical frame of reference' with 'an extraordinarily high number of remarkable, analytical, and normative, properties' (Pasinetti, 1988, p. 132).

For example, Pasinetti (1988) analyses the pre-institutional cost structure of a non-uniformly growing economy. Pasinetti constructs a 'complete generalisation of the pure labour theory of value' (Pasinetti, 1988, p. 130) by proving that, in conditions where supply equals demand, the economy's natural prices are proportional to the 'physical quantities of labour' supplied to 'vertically hyper-integrated subsystems' that include the production of net investment goods. In consequence, the labour 'embodied' in a commodity, suitably generalised, equals the labour it 'commands' in the market.

However, at the institutional stage of analysis, the 'pure labour theory of value' breaks down. Marx's 'prices of production' (Marx, [1894] 1971, ch. 9) are the steady-state prices that correspond to an institutional setup in which capitalists reallocate their capital to seek higher returns until a uniform, general rate of profit prevails across all sectors of production. Pasinetti constructs a 'complete generalisation of Marx's "transformation problem"' (Pasinetti, 1988, p. 131) by proving that, in general, production prices are not proportional to the labour supplied to the hyper-integrated subsystems. Pasinetti (1981, p. 153) concludes that 'a theory of value in terms of pure labour can never reflect the price structure that emerges from the operation of the market in a capitalist economy'.

Adam Smith restricted the labour theory to an 'early and rude state of society' that precedes the 'accumulation of stock' (Smith, [1776] 1994, p. 53). Pasinetti also restricts the explanatory reach of the labour theory, but for a different reason. According to Pasinetti, the labour theory provides a 'natural' or ideal standard from which to analyse and critique the institutional setups of actual economic systems. Pasinetti, therefore, retains an essential normative role for the labour theory.

The argument of this paper is that the labour theory of value is not merely normative but also a positive theory that applies to the price structure of a capitalist economy. I solve, or more precisely dissolve, Marx's transformation problem, and Pasinetti's generalisation of it, by extending Pasinetti's vertically integrated approach to encompass the institutional conditions of production. I construct vertically super-integrated subsystems that additionally include the material reproduction of the capitalist class as part of the subsystem. I then prove that production prices are proportional to the labour supplied to the vertically super-integrated subsystems. In consequence, the labour 'embodied' in a commodity equals the labour it 'commands', even in the circumstances of capitalist production.

Transformation problems necessarily arise when we compare institution-dependent prices with natural, or institution-independent, vertically integrated subsystems. The problems therefore dissolve once we compare prices with vertically integrated subsystems induced by the specific institutional setup of an economy. A suitably generalised labour theory of value is therefore neither incomplete or incoherent, and need not be

restricted to a normative role, but spans both the natural and institutional stages of analysis.

The structure of the paper is as follows: Sections 1 to 3 summarise Pasinetti's model and his argument for restricting the labour theory of value to a normative role. Section 4 proves that Marx's production prices are proportional to the labour supplied to the vertically super-integrated subsystems. Section 5 concludes by discussing the implications for the post Sraffian reconstruction of classical economics.

1. Hyper-subsystems and their natural prices

Sraffa (1960, p. 89) proposed to decompose an integrated economic system into 'as many parts as there are commodities in its net product, in such a way that each part forms a smaller self-replacing system the net product of which consists of only one kind of commodity. These parts we shall call "subsystems".' A subsystem is a vertically integrated 'slice' of the economy that produces a single commodity as final output and replaces the used-up means of production.

Pasinetti (1988) generalises Sraffa's approach to a growing economy where each sector produces investment goods that increase the scale of production.[2] Pasinetti defines growing 'vertically integrated hyper-subsystems' that additionally include the production of investment goods.

Pasinetti constructs an n-sector economy, which exhibits unbalanced growth, in terms of n hyper-subsystems. The total output of a hyper-subsystem is

$$\mathbf{q}_i(t) = \mathbf{q}_i(t)\mathbf{A}^\mathrm{T} + (g + r_i)\mathbf{q}_i(t)\mathbf{A}^\mathrm{T} + \mathbf{n}_i(t), \tag{1}$$

where $\mathbf{q}_i(t)$ is a vector of n quantities, $\mathbf{A} = [a_{i,j}]$ is a constant $n \times n$ input-output matrix, and $\mathbf{n}_i(t)$ is a zero vector except for the ith component, which is a scalar n_i that represents the final demand for commodity i. The total output of a hyper-subsystem, $\mathbf{q}_i(t)$, therefore breaks down into (i) replacement for used-up means of production, $\mathbf{q}_i(t)\mathbf{A}^\mathrm{T}$, (ii) additional investment in means of production, $(g + r_i)\mathbf{q}_i(t)\mathbf{A}^\mathrm{T}$, to meet increased demand for commodity i due to the growth rate, g, of the population and the per-capita growth rate, r_i, of consumption demand for commodity i (which may be positive or negative), and (iii) the final output, or net product, $\mathbf{n}_i(t)$, which is the quantity of commodity i consumed. The total labour supplied to the hyper-subsystem is

$$L_i(t) = \mathbf{l}\mathbf{q}_i^\mathrm{T}(t) = \mathbf{1}\big(\mathbf{I} - \mathbf{A}(1 + g + r_i)\big)^{-1}\mathbf{n}_i^\mathrm{T}(t), \tag{2}$$

where $\mathbf{1} = [l_i]$ is a vector of n direct labour coefficients.[3]

A hyper-subsystem includes the labour and means of production necessary for the production of its final output *and* the labour and net investment in means of production necessary for its expansion at the growth rate $(g + r_i)$. Pasinetti defines the trajectory of final demand as

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[2] For a discussion of the genesis of the concept of vertical hyper-integration, see Garbellini (2010, pp. 36–38).

[3] Equations (1) and (2) are identical to equations (2.5), (2.6) and (2.7) in Pasinetti (1988) except, in this paper, we make the simplifying assumption that $\mathbf{B} = \mathbf{I}$.

$$n_i(t) = n_i(0)e^{(g+r_i)t}, \tag{3}$$

i.e. $\frac{dn_i}{dt} = n_i(g + r_i)$, which drives the growth of the subsystem starting from its initial scale at $t = 0$. For notational convenience I now drop explicit time parameters. All subsequent algebraic statements therefore hold at an implicit time t. (I consider the implications of the trajectory of final demand in Appendix 6.4.)

Pasinetti obtains the integrated economic system by composing the n hyper-subsystems. Define the total output of the integrated economic system as the sum of its n hyper-subsystems,

$$\mathbf{q} = [q_i] = \mathbf{q}\mathbf{A}^\mathrm{T} + g\mathbf{q}\mathbf{A}^\mathrm{T} + \left(\sum_{i=1}^{n} r_i \mathbf{q}_i\right)\mathbf{A}^\mathrm{T} + \mathbf{n} \tag{4}$$

where $\mathbf{q} = \sum_{i=1}^{n} \mathbf{q}_i$, $\mathbf{n} = \sum_{i=1}^{n} \mathbf{n}_i$, and $L = \mathbf{l}\mathbf{q} = \sum_{i=1}^{n} L_i$, with standard restrictions on the eigenvalues of $\mathbf{A}$ and the feasibility of the growth rates.[4] Note that Pasinetti's model includes a steady-state economy as a special case (by setting $g = 0$ and $r_i = 0$ for all i).

Pasinetti defines natural prices that correspond to the pre-institutional stage of investigation. He stipulates that each hyper-subsystem has its own natural profit rate, $\pi_i^\star$, which 'is equal to the rate of growth of demand for the corresponding consumption good' (Pasinetti, 1988, p. 129); that is, we have n natural profit rates, $\pi_1^\star, \pi_2^\star, \ldots, \pi_n^\star$, where $\pi_i^\star = g + r_i$ and, in consequence, n vectors of natural prices, $\mathbf{p}_1, \mathbf{p}_2, \ldots, \mathbf{p}_n$, one for each hyper-subsystem, such that

$$\mathbf{p}_i = \mathbf{p}_i \mathbf{A} + \mathbf{p}_i \mathbf{A}\pi_i^\star + \mathbf{l}w, \tag{5}$$

where w is the wage rate.

Define $\mathbf{X}^{(i)}$ as the ith column and $\mathbf{X}_{(i)}$ as the ith row of $\mathbf{X}$. The price of commodity j in hyper-subsystem i therefore breaks down into (i) the cost of replacing used-up means of production, $\mathbf{p}_i \mathbf{A}^{(j)}$, (ii) the cost of net investment in additional means of production, $\mathbf{p}_i \mathbf{A}^{(j)}\pi_i^\star$, and (iii) the wage bill, $l_j w$.

In general, each commodity type has a different natural price in each hyper-subsystem. The natural profit rates 'make possible the expansion of the production of each final good according to the evolution of its final demand', which, as Bellino (2009) explains, 'provides a "social" justification for profit' in terms of the structural necessity for 'mark-up' rates that fund the growth of each hyper-subsystem. Profit and wages, in this natural system, perform different economic functions: profit is purchasing power that injects commodities back into the circular flow, for the expansion of the system, whereas wages are purchasing power that ejects commodities from the circular flow, for final consumption (Garbellini, 2010, pp. 29–30).

2. A complete generalisation of the pure labour theory of value

The concept of a vertically integrated subsystem and the real cost of production of a commodity, measured in terms of labour, are closely connected. For example, Sraffa

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[4] Equation (4) is identical to equation (2.1) in Pasinetti (1989), except, again, we set $\mathbf{B} = \mathbf{I}$.

(1960, p. 13) defines the labour cost of commodity i as the total labour supplied to the subsystem that produces a single unit of i as final output. The standard equation for classical labour-values, $\mathbf{v} = \mathbf{v}\mathbf{A} + \mathbf{1}$, immediately follows, since each labour-value, v_i, is the sum of the direct labour, l_i, supplied to sector i and the vertically integrated, indirect labour, $v_i\mathbf{A}^{(i)}$, supplied to other sectors of the economy that replace the used-up means of production (e.g. see Sraffa, 1960; Samuelson, 1971; Pasinetti, 1977).

The natural prices, $\mathbf{p}_i$, of each hyper-subsystem, given by equation (5), vary with the natural profit rate, $\pi_i^\star = g + r_i$, whereas classical labour-values do not. Classical labour-values, therefore, cannot fully explain the structure of natural prices in Pasinetti's growing economy.

In the economy defined by (4), with the set of natural prices (5), wages are the only type of income. Capitalist profit, in the sense of income received in virtue of firm ownership rather than labour supplied, is absent at the pre-institutional stage of investigation. As Reati (2000, p. 497) notes, 'the mere existence of wages could presuppose two social classes. However, on this point also Pasinetti's model is flexible, because nothing prevents us from considering a self-managed economy in which workers decide on the amount and allocation of a surplus'.

Pasinetti therefore demonstrates that capitalist profit is not the essential cause of the divergence of natural prices and classical labour-values. Even in the absence of capitalist profit, a 'transformation problem' arises: the natural prices cannot be reduced to labour-values. Capitalist social relations are therefore merely a sufficient, not a necessary, condition for the divergence of natural prices and labour-values.[5]

Pasinetti constructs a more general definition of labour cost that corresponds to his more general definition of a subsystem. The 'vertically hyper-integrated labour coefficients' generalise classical labour-values to include the labour supplied to produce net investment goods (the 'hyper-indirect' labour) as a real cost of production:

Definition 1. *Pasinetti's vertically hyper-integrated labour coefficients,* $\mathbf{v}^\star$, *are*

$$\mathbf{v}_i^\star = \mathbf{1} + \mathbf{v}_i^\star\mathbf{A} + \mathbf{v}_i^\star\mathbf{A}(g + r_i), \tag{6}$$

which is the sum of direct, indirect and 'hyper-indirect' labour.[6]

A hyper-integrated labour coefficient is therefore the total labour supplied to the hyper-subsystem. Note that, in conditions of zero growth, the hyper-integrated labour coefficients reduce to classical labour-values, i.e. $\mathbf{v}_i = \mathbf{1} + \mathbf{v}_i\mathbf{A}$.

Pasinetti then demonstrates that the natural prices of each hyper-subsystem are proportional to the vertically hyper-integrated labour coefficients:

$$\mathbf{p}_i = \mathbf{p}_i\mathbf{A} + \mathbf{p}_i\mathbf{A}\pi_i^\star + \mathbf{1}w = \mathbf{1}\big(\mathbf{I} - \mathbf{A}(1 + \pi_i^\star)\big)^{-1}w = \mathbf{v}_i^\star w.$$

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[5] This point was made, somewhat differently, by Samuelson (1971), who demonstrates that the natural prices of a post-capitalist economy, which lacks capitalist profit income, necessarily deviate from classical labour-values.

[6] Equation (6) is identical to equation (2.9) in Pasinetti (1988), except $\mathbf{B} = \mathbf{I}$.

The natural prices, therefore, reduce to the total wage bill of each hyper-subsystem, i.e. the wages of the direct, indirect and hyper-indirect labour supplied to produce unit commodities.

Pasinetti (1988, p. 130) notes that 'this is a complete generalisation of the pure labour theory of value' that recreates Smith's 'early and rude state' of society in which labour-embodied equals labour-commanded. Furthermore, 'the analytical step that allows the achievement of this result is of course a re-definition of the concept of "labour embodied", which must be intended as the quantity of labour required directly, indirectly *and* hyper-indirectly to obtain the corresponding commodity as a consumption good' (Pasinetti, 1988, pp. 131–32).

To summarise: if we relate classical labour-values to the natural prices of a hyper-subsystem, we encounter a new 'transformation problem': labour costs and nominal costs are incommensurate. This is not the classical transformation problem, but a further example of how classical labour-values cannot account for general price structures. The fundamental reason is simple: the natural prices of a hyper-subsystem include the nominal cost of net investment as a component of the price of commodities, whereas classical labour-values exclude the real cost of net investment as a component of the labour-value of commodities. In this case, the dual systems of prices and labour-values adopt different, and incommensurate, cost accounting conventions.

Pasinetti's transformation problem dissolves once we relate natural prices to the hyper-integrated labour coefficients. Hyper-integrated labour coefficients adopt the same accounting convention as the price system, and therefore include the real cost of net investment as a component of the labour-value of commodities. Commensurability is thereby restored.

Pasinetti understands that, in certain circumstances, classical labour-values undercount the total labour costs of production. He therefore constructs a more general measure of labour cost appropriate to the more general economic setting.

3. A complete generalisation of Marx's transformation problem

Pasinetti now switches to an institutional stage of investigation where capitalists, as owners of firms, receive profit income. Capitalists reallocate their capital between sectors seeking the highest returns until a general, uniform profit rate prevails across all sectors of the economy, i.e.

$$\mathbf{p} = \mathbf{pA} + \mathbf{pA}\pi + \mathbf{l}\omega. \tag{7}$$

Marx ([1894] 1971, ch. 9) called these the 'prices of production'. The production prices (7) in contrast to natural prices (5) impose a single price structure on the entire integrated economy. Also, at this institutional stage, the meaning of profit alters. Profit, in the context of capitalist property relations, is not merely a structural variable, determined by technology and growth requirements, i.e. $\mathbf{p}_i \mathbf{A}(g + r_i)$, but is now a distributional variable received in proportion to the money-capital invested in means of production within each sector of production, i.e. $\mathbf{pA}\pi$.

Pasinetti then demonstrates that a 'pure labour theory of value' breaks down in the institutional circumstances of capitalism. Pasinetti writes equation (7) in the equivalent form,

$$\mathbf{p} = \mathbf{pA} + \mathbf{pA}(g + r_i) + \mathbf{pA}(\pi - g - r_i) + \mathbf{l}w, \tag{8}$$

where $g + r_i$ is the growth rate of demand for any consumption good we care to choose (here we have chosen the ith commodity). For convenience, define the matrix $\mathbf{M}_i = \mathbf{A}\left(\mathbf{I} - \mathbf{A}(1 + g + r_i)\right)^{-1}$. We can therefore write production price equation (7) in n different, but equivalent, forms,[7]

$$\mathbf{p} = \mathbf{v}_i^\star \left(\mathbf{I} - \mathbf{M}_i(\pi - g - r_i)\right)^{-1} w, \qquad i = 1, 2, \dots, n.^{7} \tag{9}$$

Consider, for a moment, the special or accidental case, in which the general profit rate equals the growth rate of demand for commodity i; that is, $\pi = g + r_i$. Equation (9) then collapses to $\mathbf{p} = \mathbf{v}_i^\star w$ and production prices are proportional to the vertically hyper-integrated labour coefficients of hyper-subsystem i. But in general, $\pi \neq g + r_i$ for any i, and therefore production prices are not proportional to any hyper-integrated labour coefficient of any hyper-subsystem.

In fact, production prices vary independently of the vertically hyper-integrated labour coefficients because prices are a function of a global distributional variable, π, whereas the hyper-integrated labour coefficients are not. Production prices therefore cannot be reduced to labour costs, whether measured in terms of classical labour-values or Pasinetti's hyper-integrated coefficients. Pasinetti (1988, p. 131) notes, therefore, that equation (9) 'can also be regarded as providing a complete generalisation of Marx's "transformation problem"' to the case of a non-uniformly growing economy.

Pasinetti concludes, in an earlier work, that this 'analysis amounts to a demonstration that a theory of value in terms of pure labour can never reflect the price structure that emerges from the operation of the market in a capitalist economy, simply because the market is an institutional mechanism that makes proportionality to physical quantities of labour impossible to realise' (Pasinetti, 1981, p. 153).

Pasinetti therefore restricts the 'pure labour theory of value' to the pre-institutional stage of investigation, where it 'has to be taken as providing a *logical frame of reference—a conceptual construction* which defines a series, actually a family of series, of ideal *natural* prices, which possess an extraordinarily high number of remarkable, analytical, and *normative*, properties' (my emphasis) (Pasinetti, 1988, p. 132).

In the next section, I further generalise Pasinetti's vertically integrated approach. I define non-natural, or institutional subsystems, the 'vertically super-integrated subsystems', which correspond to the reproduction conditions of the specific institutional setup of capitalism. I prove that production prices are proportional to a more general measure of labour cost, the 'vertically super-integrated labour coefficients'. In consequence, the labour theory of value, suitably generalised, equally applies to the 'operation of the market in a capitalist economy'.

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[7] Equation (9) is identical to equation (4.5) in Pasinetti (1988), and we derive it from equation (8):

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\begin{aligned}
\mathbf{p} &= \mathbf{p}\mathbf{A} + \mathbf{p}\mathbf{A}(g + r_i) + \mathbf{p}\mathbf{A}(\pi - g - r_i) + \mathbf{l}w \\
\mathbf{p}(\mathbf{I} - \mathbf{A}(1 + g + r_i)) &= \mathbf{l}w + \mathbf{p}\mathbf{A}(\pi - g - r_i) \\
\mathbf{p} &= \mathbf{l}(\mathbf{I} - \mathbf{A}(1 + g + r_i))^{-1}w + \mathbf{p}\mathbf{A}(\mathbf{I} - \mathbf{A}(1 + g + r_i))^{-1}(\pi - g - r_i) \\
&= \mathbf{v}_i^\star w + \mathbf{p}\mathbf{M}_i(\pi - g - r_i).
\end{aligned}
$$

4. A general solution to the transformation problem

We first consider the special case of a steady-state economy before generalising to Pasinetti's growth model.

4.1. A special case: the steady-state economy

Production prices are a function of the distribution of nominal income between profits and wages. In order to define the 'vertically super-integrated subsystems' we require the corresponding physical data that specifies the distribution of real income. Assume, therefore, that workers receive the real wage, $\mathbf{w} = [w_i]$, and capitalists receive the consumption bundle, $\mathbf{c} = [c_i]$, such that the net product $\mathbf{n} = \mathbf{w} + \mathbf{c}$.

In conditions of zero growth, i.e. $g = 0$ and $r_i = 0$ for all i, Pasinetti's quantity equation (4) reduces to $\mathbf{q} = \mathbf{qA}^{\mathrm{T}} + \mathbf{n}$, which we expand as

$$\mathbf{q} = \mathbf{qA}^{\mathrm{T}} + \mathbf{w} + \mathbf{c}. \qquad (10)$$

We analyse the following special-case, steady-state economy:

Definition 2. *A 'steady-state economy with production prices' produces quantities, $\mathbf{q} = \mathbf{qA}^{\mathrm{T}} + \mathbf{w} + \mathbf{c}$, at prices, $\mathbf{p} = \mathbf{pA}(1+\pi) + \mathbf{l}w$, where workers and capitalists spend what they earn, $\mathbf{pw}^{\mathrm{T}} = \mathbf{lq}^{\mathrm{T}}w$ and $\mathbf{pc}^{\mathrm{T}} = \mathbf{pAq}^{\mathrm{T}}\pi$.*

In this economy, the net product is produced, distributed and consumed within the period of production. Over multiple periods, the economy self-replaces with a constant composition and scale.[8]

The production and distribution of the net product are necessarily related. For example, the quantity of commodity i consumed by worker households per unit of wage income is $w_i / \mathbf{lq}^{\mathrm{T}}w$. The income received by worker households, per unit output in sector j, is $l_j w$. Hence, consumption coefficient $w_{i,j} = w_i l_j / \mathbf{lq}^{\mathrm{T}}$ denotes the quantity of commodity i distributed to worker households per unit output of j. Define

$$\mathbf{W} = \frac{1}{\mathbf{lq}^{\mathrm{T}}}\mathbf{w}^{\mathrm{T}}\mathbf{1} = [w_{i,j}], \qquad (11)$$

as a matrix of worker consumption coefficients. $\mathbf{W}$ compactly describes the physical flow rate of consumption goods to worker households per unit outputs.

Production price equation (7) implies that profit is proportional to the money-capital 'tied up' in circulating capital, i.e. $\mathbf{pAq}^{\mathrm{T}}\pi$. The quantity of commodity i consumed by capitalist households per unit of profit income is therefore $c_i / \mathbf{pAq}^{\mathrm{T}}\pi$. The profit income received by capitalist households, per unit output in sector j, is $\mathbf{pA}^{(j)}\pi$. Hence, consumption coefficient $c_{i,j} = c_i\mathbf{pA}^{(j)} / \mathbf{pAq}^{\mathrm{T}}$ denotes the quantity of commodity i distributed to capitalist households per unit output of j. Define

$$\mathbf{C} = \frac{1}{\mathbf{pAq}^{\mathrm{T}}}\mathbf{c}^{\mathrm{T}}\mathbf{pA} = [c_{i,j}], \qquad (12)$$

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[8] Pasinetti's economy, in conditions of zero growth, reduces to a closed Leontief system with final demand equal to the consumption of workers and capitalists; see Pasinetti (1977, pp. 60–61).

as a matrix of capitalist consumption coefficients. $\mathbf{C}$ compactly describes the flow rate of consumption goods to capitalist households per unit outputs.

Arriving at matrix $\mathbf{C}$ via the price system reveals the necessary connection between capitalists' real and nominal income. Nonetheless, $\mathbf{C}$ is a 'physical' consumption matrix, which is independent of the system of prices, and solely determined by the real properties of the economy:

Proposition 1. *The capitalist consumption matrix,* $\mathbf{C}$, *is a function of the real properties of the economy, specifically the technique,* $\mathbf{A}$ *and* $\mathbf{1}$, *and the real distribution of income, given by the real wage,* $\mathbf{w}$, *and capitalist consumption bundle,* $\mathbf{c}$.

Proof. See Appendix 6.1.

For example, the price magnitudes in equation (12) cancel out yielding physical coefficients. Each coefficient $c_{i,j}$ denotes the quantity of commodity i distributed to capitalist households per unit output of commodity j. The technique $\mathbf{A}$ and consumption matrices, $\mathbf{W}$ and $\mathbf{C}$, together specify the physical flow rates of goods between sectors of production and households.

Recall that a subsystem is a 'self-replacing system' that replaces used-up means of production and produces a final output. A Sraffian subsystem, for example, is the direct and indirect production that produces a single component of the *net product* as final output, where the net product consists of consumption goods. All consumption goods, in a Sraffian subsystem, are final outputs or 'surplus', and therefore not replaced by the subsystem.

A given economic system, however, can be decomposed into alternative kinds of subsystems. Define a 'vertically super-integrated subsystem' as the direct, indirect and 'super-indirect' production that produces a single component of the *real wage* as final output, where 'super-indirect' refers to the production of capitalist consumption goods. Note that super-integrated subsystems, in contrast to Pasinetti's subsystems, now include the reproduction conditions of a specific institutional setup. In a super-integrated subsystem, we only consider the real wage of workers as the final output or 'surplus'. In consequence, a super-integrated subsystem also replaces the real income of capitalists. More formally, the total output of the ith vertically super-integrated subsystem is

$$\hat{\mathbf{q}}_i = \hat{\mathbf{q}}_i\mathbf{A}^{\mathrm{T}} + \hat{\mathbf{q}}_i\mathbf{C}^{\mathrm{T}} + \mathbf{w}_i,$$

where $\mathbf{w}_i$ is a zero vector except for the ith component that equals w_i, which is the wage demand for commodity i. A super-integrated subsystem additionally vertically integrates the production of capitalist consumption goods. The capitalist consumption matrix, $\mathbf{C}$, therefore appears as a real cost of production. The total output of the steady-state economy is then the composition of the vertically super-integrated subsystems, i.e. $\mathbf{q} = \sum \hat{\mathbf{q}}_i$.

Sraffa's and Pasinetti's natural subsystems are defined by technological and accumulation conditions alone. In contrast, the super-integrated subsystems are also defined by social and institutional conditions. A super-integrated subsystem captures the institutional fact that production, in a capitalist system, materially reproduces a capitalist class at a given level of real income.

A 'vertically super-integrated labour coefficient', denoted $\tilde{v}_i$, is the total labour supplied to the ith super-integrated subsystem when it produces a unit component of the real wage as final output, i.e. when $w_i = 1$. For clarity, we now calculate this quantity step-by-step.

Consider the production of 1 unit of commodity i in super-integrated subsystem i. How much labour does this production require? It requires l_i units of direct labour, $\mathbf{1A}^{(i)}$ units of indirect labour and $\mathbf{1C}^{(i)}$ units of super-indirect labour, giving a total of $\mathbf{1(A}^{(i)} + \mathbf{C}^{(i)})$ units of labour operating in parallel to produce the output, replace used-up means of production and replace capitalist consumption goods, respectively. Define $\widetilde{\mathbf{A}} = \mathbf{A} + \mathbf{C}$ as the technique augmented by capitalist consumption. Matrix $\widetilde{\mathbf{A}}$ compactly represents the commodities used up during the production of each commodity type *including* the commodities consumed by capitalists. The sum of direct, indirect and super-indirect labour is then $l_i + \mathbf{1}\widetilde{\mathbf{A}}^{(i)}$.

However, the indirect and super-indirect production itself uses up means of production and consumption goods, specifically the bundle $\widetilde{\mathbf{A}}\widetilde{\mathbf{A}}^{(i)}$, which is contemporaneously replaced by the supply of additional labour, $\mathbf{1}\widetilde{\mathbf{A}}\widetilde{\mathbf{A}}^{(i)}$. To count all the direct, indirect and super-indirect labour, we must continue the sum; that is,

$$\tilde{v}_i = l_i + \mathbf{1}\widetilde{\mathbf{A}}^{(i)} + \mathbf{1}\widetilde{\mathbf{A}}\widetilde{\mathbf{A}}^{(i)} + \mathbf{1}\widetilde{\mathbf{A}}^2\widetilde{\mathbf{A}}^{(i)} + \dots$$

$$= l_i + \mathbf{1}\left(\sum_{n=0}^{\infty}\widetilde{\mathbf{A}}^n\right)\widetilde{\mathbf{A}}^{(i)}.$$

This sum represents the total labour supplied to the ith super-integrated subsystem when it produces 1 unit as final output.

The vector $\tilde{\mathbf{v}}$ of super-integrated coefficients is therefore $\tilde{\mathbf{v}} = \mathbf{1} + \mathbf{1}\left(\sum_{n=0}^{\infty}\widetilde{\mathbf{A}}^n\right)\widetilde{\mathbf{A}} = \mathbf{1}\sum_{n=0}^{\infty}\widetilde{\mathbf{A}}^n$. Assuming that capitalist consumption is feasible, given the technology, then matrix $\widetilde{\mathbf{A}}$ is productive, and we may replace the infinite series with the Leontief inverse, $\mathbf{1}\sum_{n=0}^{\infty}\widetilde{\mathbf{A}}^n = \mathbf{1}(\mathbf{I} - \widetilde{\mathbf{A}})^{-1}$; in consequence:

Definition 3. *The 'vertically super-integrated labour coefficients', $\tilde{\mathbf{v}}$, in a steady-state economy with production prices, are*

$$\tilde{\mathbf{v}} = \mathbf{1} + \tilde{\mathbf{v}}\widetilde{\mathbf{A}}$$

$$= \mathbf{1} + \tilde{\mathbf{v}}\mathbf{A} + \tilde{\mathbf{v}}\mathbf{C}, \tag{13}$$

which is the sum of direct, indirect and super-indirect labour costs.

The definition of the super-integrated coefficients does not provide or rely upon any theory of income distribution or profit. However, in order to calculate the super-integrated coefficients, the distribution of real income must be a given datum, in the same manner that, in order to calculate production prices, the distribution of nominal income must be a given datum. Conjectural variation of either the real or nominal distribution of income then affects both the super-integrated coefficients and production prices.

The super-integrated labour coefficients, although more complex than classical labour-values, nonetheless directly relate, in a straightforward manner, to the labour supplied during the production period.

For example, Pasinetti (1980, p. 21) classifies the total labour supplied in two ways: as (i) the sum of direct labour supplied to each sector of production, $\sum l_i q_i = \mathbf{lq}^{\mathrm{T}}$, or (ii) the sum of direct and indirect labour supplied to each Sraffian subsystem, $\sum v_i n_i = \mathbf{vn}^{\mathrm{T}}$. The classifications are quantitatively equal, that is, $\mathbf{lq}^{\mathrm{T}} = \mathbf{vn}^{\mathrm{T}}$, because the Sraffian subsystems collectively produce the net product as final output and exhaust the total supplied labour:

Proposition 2. *The total labour supplied equals the classical labour-value of the net product,* $\mathbf{lq}^{\mathrm{T}} = \mathbf{vn}^{\mathrm{T}}$.

Proof. From (10), $\mathbf{q} = \mathbf{qA}^{\mathrm{T}} + \mathbf{n} = \mathbf{n}(\mathbf{I} - \mathbf{A}^{\mathrm{T}})^{-1}$. Hence $\mathbf{lq}^{\mathrm{T}} = (\mathbf{I} - \mathbf{A})^{-1}\mathbf{n}^{\mathrm{T}} = \mathbf{vn}^{\mathrm{T}}$.

The super-integrated subsystems provide another partition of the same economy. The total labour supplied can also be classified as (iii) the sum of direct, indirect and super-indirect labour supplied to each super-integrated sector, $\sum \tilde{v}_i w_i = \tilde{\mathbf{v}}\mathbf{w}^{\mathrm{T}}$. Again, this classification is quantitatively equal to the total labour supplied, that is, $\mathbf{lq}^{\mathrm{T}} = \tilde{\mathbf{v}}\mathbf{w}^{\mathrm{T}}$, because the super-integrated subsystems collectively produce the real wage as final output and exhaust the total supplied labour:

Proposition 3. *The total labour supplied equals the super-integrated labour-value of the real wage,* $\tilde{\mathbf{v}}\mathbf{w}^{\mathrm{T}} = \mathbf{lq}^{\mathrm{T}}$.

Proof. From (12), $\mathbf{Cq}^{\mathrm{T}} = (1/\mathbf{pAq}^{\mathrm{T}})\mathbf{c}^{\mathrm{T}}\mathbf{pAq}^{\mathrm{T}} = \mathbf{c}^{\mathrm{T}}$. Substitute into (10) to yield, $\mathbf{q} = \mathbf{qA}^{\mathrm{T}} + \mathbf{qC}^{\mathrm{T}} + \mathbf{w} = \mathbf{q}\widetilde{\mathbf{A}} + \mathbf{w} = \mathbf{w}(\mathbf{I} - \widetilde{\mathbf{A}})^{-1}$. Hence, $\mathbf{lq}^{\mathrm{T}} = (\mathbf{I} - \widetilde{\mathbf{A}})^{-1}\mathbf{w}^{\mathrm{T}} = \tilde{\mathbf{v}}\mathbf{w}^{\mathrm{T}}$.

Now that we've defined the super-integrated coefficients, we can relate them to production prices.

Theorem 1. *The production prices of a steady-state economy are proportional to the super-integrated labour coefficients,* $\mathbf{p} = \tilde{\mathbf{v}}w$.

Proof. Since capitalists spend what they earn, $\mathbf{pAq}^{\mathrm{T}}\pi = \mathbf{pc}^{\mathrm{T}}$. Substitute for π into price equation (7): $\mathbf{p} = \mathbf{pA}(1 + \dfrac{\mathbf{pc}^{\mathrm{T}}}{\mathbf{pAq}^{\mathrm{T}}}) + \mathbf{l}w = \mathbf{pA} + \dfrac{\mathbf{pc}^{\mathrm{T}}}{\mathbf{pAq}^{\mathrm{T}}}\mathbf{pA} + \mathbf{l}w = \mathbf{p}(\mathbf{A} + \dfrac{1}{\mathbf{pAq}^{\mathrm{T}}}\mathbf{c}^{\mathrm{T}}\mathbf{pA}) + \mathbf{l}w = \mathbf{pA} + \mathbf{pC} + \mathbf{l}w = \mathbf{p}\widetilde{\mathbf{A}} + \mathbf{l}w = \mathbf{l}(\mathbf{I} - \widetilde{\mathbf{A}})^{-1}w = \tilde{\mathbf{v}}w$.

Production prices equal the total wage bill of each super-integrated subsystem, i.e. the wages of the direct, indirect and super-indirect labour supplied to produce unit commodities. This more general definition of labour costs replicates the result, established by Adam Smith for an 'early and rude state' of society, that 'labour embodied' equals 'labour commanded'.

The 'physical' configuration of the steady-state economy, specifically the prevailing technique and distribution of real income, determines both the structure of the vertically super-integrated labour coefficients and the structure of production prices. Recall that Marx's transformation problem arises because production prices vary with the distribution of income but classical labour-values do not. The super-integrated coefficients, in contrast, also vary with the distribution of income because they vertically integrate over the production of the real income of capitalists. In consequence, production prices and labour costs, suitably measured, are necessarily dual to each other and 'two sides of the same coin'.

The conceptual advance, compared to prior classical analysis, is to consider not only technical but also social conditions of reproduction. A capitalist economy, for instance, functions according to specific distributional rules, such as wage income from labour, and profit income from capital. By further generalising vertical integration to include the social conditions of reproduction, we 'embed' the distributional rules of an economy within super-integrated subsystems. We can then compute the real costs of production caused by specific institutional setups and compare those real costs to the system of prices caused by the same institutional setup.

Theorem 1 reveals the essential duality between real costs of production, measured in labour time, and the natural prices of a capitalist economy. In consequence, the value theory of classical political economy, rather than being restricted to special cases, has broad applicability and important implications for social theory as a whole.

Next, I generalise this result to Pasinetti's growth model. The generalisation does not require any new arguments or ideas. However, the super-integrated coefficients, in the case of non-uniform growth, include both hyper and super-indirect labour.

4.2. The general case: Pasinetti's non-uniform growth model

In the more general circumstances of non-uniform growth, the final demand is variable. The net product is therefore a function of time, i.e. $\mathbf{n}(t) = \mathbf{w}(t) + \mathbf{c}(t)$. Assume an initial distribution of real income, $\mathbf{w}(0)$ and $\mathbf{c}(0)$. The trajectory of final demand, from equation (3), is then $\mathbf{n}(t) = [w_i(0)e^{(g+r_i)t}] + [c_i(0)e^{(g+r_i)t}]$. The vectors $\mathbf{w}$ and $\mathbf{c}$ now implicitly refer to time-varying consumption bundles that, following Pasinetti, drive the growth of the economy.[9]

For notational convenience, define the 'non-uniform capital investment vector', $\mathbf{g} = \sum_{i=1}^{n} r_i \mathbf{q}_i \mathbf{A}^{\mathrm{T}} = [g_i]$, where $\mathbf{q}_i = [q_{i,j}]$ is the total output of hyper-subsystem i as defined by equation (1), and each g_i is the quantity of commodity i produced as additional means of production, in the economy as a whole, in order to meet the total non-uniformly growing demand.[10] Let $\Gamma = \mathbf{diag}(\mathbf{g})\mathbf{diag}(\mathbf{q})^{-1} = [\lambda_{i,j}]$ be a diagonal 'non-uniform capital investment matrix', where each element on the diagonal, $\lambda_{i,i} = g_i / q_i$, is the quantity of i produced as additional means of production, per unit output, to meet the total non-uniformly growing demand (and $\lambda_{i,j} = 0$ for $i \neq j$). Rewrite Pasinetti's quantity equation (4) in the equivalent form,

$$\mathbf{q} = \mathbf{q}\mathbf{A}^{\mathrm{T}}(1+g) + \mathbf{q}\Gamma + \mathbf{w} + \mathbf{c}. \tag{14}$$

The total profit income remains $\mathbf{p}\mathbf{A}\mathbf{q}^{\mathrm{T}}\pi$, as in the simpler case of a steady-state economy. But now a fraction of profit is invested in additional means of production to satisfy increased demand: $\mathbf{p}\mathbf{A}\mathbf{q}^{\mathrm{T}}g$ is invested to satisfy the increase in demand due to population growth, and $\mathbf{p}\Gamma\mathbf{q}^{\mathrm{T}}$ is invested to satisfy the non-uniform change in demand. The residual profit that remains for capitalists to spend on personal consumption is therefore $Y = \mathbf{p}(\mathbf{A}(\pi - g) - \Gamma)\mathbf{q}^{\mathrm{T}}$.

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[9] Of course, in the context of an actual capitalist economy, rather than Pasinetti's system, growth is not driven by exogenous real demand. My goal here is to explore the full implications of Pasinetti's imposition of a 'capitalist' price structure on his model rather than develop a realistic growth model of capitalism.

[10] Vector $\mathbf{g}$ features in Pasinetti's equation (4) that defines the total output of the integrated economy.

The full specification of Pasinetti's non-uniformly growing economy with production prices is therefore:

Definition 4. *A 'non-uniformly growing economy with production prices' produces quantities,* $\mathbf{q} = \mathbf{qA}^{\mathrm{T}}(1+g) + \mathbf{q\Gamma} + \mathbf{w} + \mathbf{c}$, *at prices,* $\mathbf{p} = \mathbf{pA}(1+\pi) + \mathbf{l}w$, *where workers and capitalists spend what they earn,* $\mathbf{pw}^{\mathrm{T}} = \mathbf{lq}^{\mathrm{T}}w$ *and* $\mathbf{pc}^{\mathrm{T}} = \mathbf{p}(\mathbf{A}(\pi - g) - \mathbf{\Gamma})\mathbf{q}^{\mathrm{T}}$.

The production and distribution of the net product are, once again, necessarily related. The matrix of worker consumption coefficients, $\mathbf{W}$, is unchanged from the steady-state case. However, the matrix of capitalist consumption coefficients, $\mathbf{C}$, differs because capitalists invest a fraction of their profit income in additional means of production. The quantity of commodity i consumed by capitalists per unit of residual profit income is now c_i / Y. The residual profit received, per unit output in sector j, is $\mathbf{p}(\mathbf{A}^{(j)}(\pi - g) - \mathbf{\Gamma}^{(j)})$. Hence, consumption coefficient $c_{i,j} = c_i \mathbf{p}(\mathbf{A}^{(j)}(\pi - g) - \mathbf{\Gamma}^{(j)}) / Y$ denotes the quantity of commodity i distributed to capitalists per unit output of j. Define

$$\mathbf{C} = \frac{1}{\mathbf{p}(\mathbf{A}(\pi - g) - \mathbf{\Gamma})\mathbf{q}^{\mathrm{T}}} \mathbf{c}^{\mathrm{T}}\mathbf{p}(\mathbf{A}(\pi - g) - \mathbf{\Gamma}) = [c_{i,j}]$$

(15)

as the matrix of capitalist consumption coefficients. (Note that, when $g = 0$ and $r_i = 0$ for all i, this definition of $\mathbf{C}$ reduces to the definition for the steady-state economy.)

Pasinetti's hyper-integrated subsystems include the direct, indirect and hyper-indirect production that produces a single component of the net product as final output. A vertically super-integrated subsystem, in the context of non-uniform growth, is the direct, indirect, hyper and super-indirect production that produces a single component of the real wage as final output. The total output of the ith vertically super-integrated subsystem is

$$\hat{\mathbf{q}}_i = \hat{\mathbf{q}}_i\mathbf{A}^{\mathrm{T}} + \hat{\mathbf{q}}_i\mathbf{A}^{\mathrm{T}}g + \hat{\mathbf{q}}_i\mathbf{\Gamma} + \hat{\mathbf{q}}_i\mathbf{C}^{\mathrm{T}} + \mathbf{w}_i,$$

where $\mathbf{w}_i$ is defined as before and $\mathbf{q} = \sum \hat{\mathbf{q}}_i$.

The net investment in a hyper-integrated subsystem, from equation (1), is $\mathbf{q}_i\mathbf{A}^{\mathrm{T}}(g + r_i)$, which is independent of the cross-demand effects of the non-uniform growth of the other hyper-integrated subsystems (i.e. the r_j for all $j \neq i$). In contrast, the net investment in a super-integrated subsystem, $\hat{\mathbf{q}}_i\mathbf{A}^{\mathrm{T}}g + \hat{\mathbf{q}}_i\mathbf{\Gamma}$, includes cross-demand effects.

The technique augmented by hyper and super-indirect real costs of production is then

$$\widehat{\mathbf{A}} = \mathbf{A} + \mathbf{A}g + \mathbf{\Gamma} + \mathbf{C}.$$

Matrix $\widehat{\mathbf{A}}$ compactly represents the commodities used up during the production of each commodity type, including the production of net investment goods and capitalist consumption goods.

Definition 5. *The 'vertically super-integrated labour coefficients',* $\hat{\mathbf{v}}$, *in a non-uniformly growing economy with production prices, are*

$$\hat{\mathbf{v}} = \mathbf{1} + \hat{\mathbf{v}}\widehat{\mathbf{A}}$$

$$= \mathbf{1} + \hat{\mathbf{v}}\mathbf{A} + \hat{\mathbf{v}}(\mathbf{A}g + \mathbf{\Gamma}) + \hat{\mathbf{v}}\mathbf{C}, \tag{16}$$

which is the sum of direct, indirect, hyper and super-indirect labour costs.[11]

The hyper and super-integrated coefficients are identical in circumstances of zero non-uniform growth and zero capitalist consumption. And both the hyper and super-integrated coefficients reduce to the classical definition of labour-value in circumstances of zero growth and zero capitalist consumption (i.e. Smith's 'early and rude state'). As before, the super-integrated subsystems collectively produce the real wage as final output and exhaust the total supplied labour; in consequence, $\mathbf{l}\mathbf{q}^{\mathrm{T}} = \hat{\mathbf{v}}\mathbf{w}^{\mathrm{T}}$.

We now state the main result of the paper:

Theorem 2. The production prices of a non-uniformly growing economy are proportional to the super-integrated labour coefficients, $\mathbf{p} = \hat{\mathbf{v}}w$.

Proof. From (15), $\mathbf{p}\mathbf{C} = (1/Y)\mathbf{p}\mathbf{c}^{\mathrm{T}}\mathbf{p}(\mathbf{A}(\pi - g) - \mathbf{\Gamma})$. Hence, $\mathbf{p}(\mathbf{A}(\pi - g) - \mathbf{\Gamma}) = (Y/\mathbf{p}\mathbf{c}^{\mathrm{T}})\mathbf{p}\mathbf{C}$. Write price equation (7) in the equivalent form, $\mathbf{p} = \mathbf{p}\mathbf{A} + \mathbf{p}(\mathbf{A}g + \mathbf{\Gamma}) + \mathbf{p}(\mathbf{A}(\pi - g) - \mathbf{\Gamma}) + \mathbf{l}w$, and then substitute to yield $\mathbf{p} = \mathbf{p}\mathbf{A} + \mathbf{p}(\mathbf{A}g + \mathbf{\Gamma}) + (Y/\mathbf{p}\mathbf{c}^{\mathrm{T}})\mathbf{p}\mathbf{C} + \mathbf{l}w$. Since capitalists spend what they earn, $\mathbf{p}\mathbf{c}^{\mathrm{T}} = Y$. Hence, $\mathbf{p} = \mathbf{p}\mathbf{A} + \mathbf{p}(\mathbf{A}g + \mathbf{\Gamma}) + \mathbf{p}\mathbf{C} + \mathbf{l}w = \mathbf{p}\widehat{\mathbf{A}} + \mathbf{l}w = \mathbf{1}(\mathbf{I} - \widehat{\mathbf{A}})^{-1}w = \hat{\mathbf{v}}w$.

Production prices, in Pasinetti's non-uniformly growing economy, equal the total wage bill of each super-integrated subsystem, i.e. the wages of the direct, indirect, hyper and super-indirect labour supplied to reproduce unit commodities. Once again, a more general definition of labour costs replicates Adam Smith's result that 'labour embodied' equals 'labour commanded'.[12]

Theorem 2, it should be emphasised, undermines the logical basis for any claim that a labour theory of value is incoherent because production prices and labour-values are quantitatively incommensurate in linear production models (e.g. Samuelson, 1971; Lippi, 1979; Steedman, 1981).

4.3. Technical and social cost structures

Marx's transformation problem, and Pasinetti's generalisation, reduce to mismatches between production prices and labour costs. Production prices include institutional or social costs, specifically a profit rate that includes the income of a capitalist class. In contrast, classical labour-values, and Pasinetti's hyper-integrated labour coefficients, are purely technical costs of production, and therefore ignore the real cost of producing capitalist income. Transformation problems necessarily arise when we contravene Pasinetti's separation thesis and compare a nominal cost structure that belongs to an institutional stage of analysis with a real cost structure that belongs to a natural, or pre-institutional, stage of analysis. A commensurate relationship cannot obtain between cost structures defined by incommensurate accounting conventions.

Pasinetti recognises the need to extend the classical theory in order to explain the structure of natural price systems. He constructs more general measures of labour cost

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[11] Note that definition 5 reduces to definition 3 in conditions of zero growth.
[12] Note that Theorem 2 reduces to Theorem 1 in conditions of zero growth.

that take into account additional features of the circumstances of production, such as the labour cost of net capital investment. Despite this important conceptual advance, Pasinetti nonetheless believes, in virtue of the transformation problems, that a labour theory of value 'can never reflect the price structure that emerges from the operation of the market in a capitalist economy' (Pasinetti, 1981, p. 153).

Theorems 1 and 2, by generalising the vertically integrated approach to encompass social and institutional conditions, demonstrate the contrary. Production prices, in both steady-state and non-uniformly growing economies, are proportional to physical quantities of labour, the vertically super-integrated labour coefficients, which include the additional labour supplied to produce the real income of capitalists. The super-integrated labour coefficients capture the real cost structure that 'emerges from the operation of the market in a capitalist economy' (Pasinetti, 1981, p. 153). The transformation problems therefore dissolve once we observe Pasinetti's separation thesis and compare the nominal and real cost structures that manifest at the same, institutional stage of analysis.

5. Conclusion

Pasinetti's separation thesis, and his generalisation of the vertically integrated approach, are powerful analytic devices. However, Pasinetti's specific proposal to restrict the labour theory of value to a normative role, a kind of 'logical frame of reference', is unnecessary. Pasinetti's theoretical innovations instead point in the opposite direction and toward a full generalisation of the classical labour theory and its reinstatement as the foundational theory of value for economic analysis within the 'production paradigm' (Pasinetti, 1986). The more general labour theory spans both the natural and institutional stages of analysis and therefore can address normative issues in the critique of political economy, and positive issues in the analysis of specific economic systems.

The more general theory, sketched here in an initial and preliminary manner, admits both technical and social measures of labour cost and applies both kinds of measures in the appropriate contexts. For example, in this more general framework, classical labour-values apply to distribution-independent questions about an economy, such as measuring the technical productivity of labour (e.g. Flaschel, 2010, part 1) or the surplus-labour supplied by workers (e.g. Marx, [1867] 1954), whereas the super-integrated labour coefficients apply to distribution-dependent questions, such as the relationship between relative prices and the actual labour time supplied to produce commodities, i.e. issues in the theory of value.

The post-Sraffian separation of the classical surplus approach to income distribution from its labour theory of value does not constitute a sophisticated rejection of naive 'substance' theories of value but indicates a failure to resolve the classical contradictions, such as Marx's transformation problem. The separation ultimately derives from the classical error of comparing technical with social cost structures (see also Wright, 2014, 2015). The post-Sraffian reconstruction of classical economics therefore dispenses with an essential aim of a theory of economic value, which is to explain what the unit of account might measure or refer to. Theorems 1 and 2, which demonstrate that production prices are proportional to physical quantities of labour, start to put the pieces back together again.

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6. Appendix

Here, for clarity, I also include numerical examples of Theorems 1 and 2 for an $n = 2$ economy, followed by a brief discussion of time-varying final demand.

6.1. Proof of Proposition 1

Lemma 1. *In a steady-state economy with production prices, the rate of profit, π, is the dominant eigenvalue of matrix $(\mathbf{I} - \mathbf{A} - \mathbf{W})\mathbf{A}^{-1}$.*

Proof. Let $\mathbf{X} = (\mathbf{I} - \mathbf{A} - \mathbf{W})\mathbf{A}^{-1}$. Then $\mathbf{pX} = \lambda\mathbf{p}$ is an eigenvalue equation. Solve the characteristic equation, $\det(\mathbf{X} - \lambda\mathbf{I}) = 0$, to obtain $\lambda = \lambda^*$, where λ^* is the Perron-Frobenius root (dominant eigenvalue). Multiple the eigenvalue equation by $\mathbf{A}$ to obtain $\lambda^*\mathbf{pA} = \mathbf{p} - \mathbf{pA} - \mathbf{pW}$, which implies $\mathbf{p} = \mathbf{pA}(1 + \lambda^*) + \mathbf{pW}$. Since $\mathbf{W} = (1/\mathbf{lq}^{\mathrm{T}})\mathbf{w}^{\mathrm{T}}\mathbf{1}$, then $\mathbf{p} = \mathbf{pA}(1 + \lambda^*) + \mathbf{l}w$, which are the production prices of a steady-state economy. Hence, $\pi = \lambda^*$.

Proposition 1. *The capitalist consumption matrix, $\mathbf{C}$, is a function of the real properties of the economy, specifically the technique, $\mathbf{A}$ and $\mathbf{1}$, and the real distribution of income, given by the real wage, $\mathbf{w}$, and capitalist consumption bundle, $\mathbf{c}$.*

Proof. Define the capitalist consumption matrix,

$$\mathbf{C} = \frac{\mathbf{c}^{\mathrm{T}}\mathbf{1}(\mathbf{I} - \mathbf{A}(1 + \pi))^{-1}\mathbf{A}}{\mathbf{1}(\mathbf{I} - \mathbf{A}(1 + \pi))^{-1}\mathbf{A}(\mathbf{w} + \mathbf{c})^{\mathrm{T}}(\mathbf{I} - \mathbf{A})^{-1}}.$$

Note that the profit rate, π, by lemma 1, is a function of $\mathbf{A}$ and worker consumption matrix $\mathbf{W}$. Matrix $\mathbf{W}$, by equation (11), is a function of $\mathbf{l}$, $\mathbf{w}$ and quantities, $\mathbf{q}$. And $\mathbf{q}$, by equation (10), is a function of $\mathbf{A}$, $\mathbf{w}$ and $\mathbf{c}$. Hence, $\mathbf{C}$, as defined in equation (17), is a function of the real properties of the economy only.

We now show that definition (17) is equivalent to definition (12). Note that $\mathbf{p} = \mathbf{1}(\mathbf{I} - \mathbf{A}(1 + \pi))^{-1}$ (from equation (7)) and $\mathbf{q}^{\mathrm{T}} = (\mathbf{w} + \mathbf{c})^{\mathrm{T}}(\mathbf{I} - \mathbf{A})^{-1}$ (from equation (17)). Substitute into equation (17) to yield $\mathbf{C} = \mathbf{c}^{\mathrm{T}}\mathbf{pA}/\mathbf{pAq}^{\mathrm{T}}$.

6.2. Numerical example of Theorem 1

The following proposition, which links the technique, capitalist consumption and the profit rate, will be useful:

Proposition 4. *In a steady-state economy with production prices, $\mathrm{Tr}(\mathbf{CA}^{-1}) = \pi$.*

Proof. Since capitalists spend what they earn, $\mathbf{pAq}^{\mathrm{T}}\pi = \mathbf{pc}^{\mathrm{T}}$. Hence, $1/\mathbf{pAq}^{\mathrm{T}} = \pi/\mathbf{pc}^{\mathrm{T}}$. From equation (12), $\mathbf{CA}^{-1} = (1/\mathbf{pAq}^{\mathrm{T}})\mathbf{c}^{\mathrm{T}}\mathbf{p}$. Substitute for $1/\mathbf{pAq}^{\mathrm{T}}$ to yield $\mathbf{CA}^{-1} = (\pi/\mathbf{pc}^{\mathrm{T}})\mathbf{c}^{\mathrm{T}}\mathbf{p}$. An inner product is the trace of its outer product, i.e. $\mathrm{Tr}(\mathbf{c}^{\mathrm{T}}\mathbf{p}) = \mathbf{pc}^{\mathrm{T}}$. Hence, $\mathrm{Tr}(\mathbf{CA}^{-1}) = \pi$.

Consider a stationary economy with (i) technique, $\mathbf{A} = \begin{bmatrix} 0 & 0.5 \\ 0.25 & 0 \end{bmatrix}$ and $\mathbf{1} = \begin{bmatrix} 1 & 2 \end{bmatrix}$

and (ii) capitalist consumption matrix, $\mathbf{C} = \begin{bmatrix} 0 & 0 \\ 0.005 & 0.006 \end{bmatrix}$.

The technique and capitalist consumption matrix determine the super-integrated labour coefficients, i.e. from definition 3, $\tilde{\mathbf{v}} = \mathbf{1}(\mathbf{I} - (\mathbf{A} + \mathbf{C}))^{-1} = \begin{bmatrix} 1.74 & 2.89 \end{bmatrix}$.

The technique and capitalist consumption matrix determine the profit rate, i.e. from Proposition 4, $\pi = \mathrm{Tr}\left(\begin{bmatrix} 0 & 0 \\ 0.012 & 0.02 \end{bmatrix}\right) = 0.02$. The profit rate determines production prices, i.e. from equation (7), $\mathbf{p} = \mathbf{l}(\mathbf{I} - \mathbf{A}(1+\pi))^{-1}w = [1.74 \quad 2.89]w$.

Hence, $\mathbf{p} = \tilde{\mathbf{v}}w$, as per Theorem 1.

6.3. Numerical example of Theorem 2

The following proposition, which links the technique, non-uniform growth, capitalist consumption and the profit rate, will be useful:

Proposition 5. *In a non-uniformly growing economy with production prices,*

$$\mathrm{Tr}\left(\mathbf{C}\left(\mathbf{A}(\pi - g) - \mathbf{\Gamma}\right)^{-1}\right) = 1.$$

Proof. Since capitalists spend what they earn, hence $1/Y = 1/\mathbf{pc}^{\mathrm{T}}$. From equation (15), $\mathbf{C}(\mathbf{A}(\pi - g) - \mathbf{\Gamma})^{-1} = (1/Y)\mathbf{c}^{\mathrm{T}}\mathbf{p}$. Substitute for $1/Y$ to yield $\mathbf{C}(\mathbf{A}(\pi - g) - \mathbf{\Gamma})^{-1} = (1/\mathbf{pc}^{\mathrm{T}})\mathbf{c}^{\mathrm{T}}\mathbf{p}$. An inner product is the trace of its outer product, i.e. $\mathrm{Tr}(\mathbf{c}^{\mathrm{T}}\mathbf{p}) = \mathbf{pc}^{\mathrm{T}}$. Hence, $\mathrm{Tr}(\mathbf{C}(\mathbf{A}(\pi - g) - \mathbf{\Gamma})^{-1}) = 1$.

Consider a non-uniformly growing economy at time t with (i) technique, $\mathbf{A} = \begin{bmatrix} 0 & 0.5 \\ 0.25 & 0 \end{bmatrix}$ and $\mathbf{l} = [1 \quad 2]$, (ii) capitalist consumption matrix, $\mathbf{C} = \begin{bmatrix} 0 & 0 \\ 0.0063 & 0.00088 \end{bmatrix}$, (iii) $g = 0.01$ and (iv) $\mathbf{\Gamma} = \begin{bmatrix} 0.00095 & 0 \\ 0 & 0.0074 \end{bmatrix}$.

This is sufficient information to compute the super-integrated labour-values. From definition 5, $\hat{\mathbf{v}} = \mathbf{l}(\mathbf{I} - \mathbf{A}(1+g) - \mathbf{\Gamma} - \mathbf{C})^{-1} = [1.75 \quad 2.91]$.

From Proposition 5, $\mathrm{Tr}(\mathbf{C}(\mathbf{A}(\pi - g) - \mathbf{\Gamma})^{-1}) = 1$. Solve to yield the profit rate, $\pi = 0.037$. Production prices, from equation (7), are then $\mathbf{p} = \mathbf{l}(\mathbf{I} - \mathbf{A}(1+\pi))^{-1}w = [1.75 \quad 2.91]w$.

Hence, $\mathbf{p} = \hat{\mathbf{v}}w$, as per Theorem 2.

6.4. The trajectory of final demand

Final demand, $\mathbf{n}(t)$, grows exponentially, such that

$$\mathbf{n}(t) = \mathbf{w}(t) + \mathbf{c}(t) = [w_i(0)e^{(g+r_i)t}] + [c_i(0)e^{(g+r_i)t}]$$

(see Section 4.2). In general, given non-uniform growth, that is, $r_i \neq r_j$ for some i and j, then matrices $\mathbf{\Gamma}$ and $\mathbf{C}$ and the profit rate, π, are not constant but vary with time. In consequence, both production prices, $\mathbf{p}$, and the vertically super-integrated labour coefficients, $\hat{\mathbf{v}}$, vary along the growth trajectory. Nonetheless, Theorem 2 holds at every instant of time. In consequence, the proportionality of production prices and super-integrated labour-values is a time-invariant property of Pasinetti's model.