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The Transformation of Values into Prices of Production in Marx's Scheme of Expanded Reproduction

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Abstract

This paper analyzes the formation of a general rate of profit and the subsequent prices of production in the context of Marx's two-sector scheme of expanded reproduction. We show that a consistent solution of the transformation problem can be derived by incorporating the original transformation procedure into the inter-temporal equilibrium framework provided by the scheme of accumulation. Previous solutions for the special case of simple reproduction are also examined, and it is shown that all these solutions rest on the restrictive assumption of constant wage rates.

JEL Classification: B14, B16, E11, E20, E21, E22, P16, P17

Keywords

transformation problem, scheme of reproduction, accumulation of capital, Karl Marx

1. Introduction

The main purpose of this paper is to identify the system of equations underlying Marx's scheme of expanded reproduction and, on this basis, to analyze the consequences of relaxing the original assumption that commodities are exchanged according to their values instead of their prices of production. As a result of this analysis, we are able to provide an internally consistent solution of the so-called *transformation problem*, which allows us to assess the influence of the formation of a general (uniform) rate of profit on the behavior of Marx's model. In strong contrast with the long tradition in the analysis of this subject, the solution provided in this paper does not rest on the restrictive assumption of constant (nominal and/or real) wage rates, as adopted by the bulk of related literature.

The crudest version of this assumption was originally proposed by Ladislaus von Bortkiewicz (1984: 212): "For the sake of simplicity it is assumed that the capitalists advance consumption goods *in natura* so that the workers take no direct part in commodity exchanges." The crucial role of such a simplification was incisively stressed by Paul Samuelson (1970: 425) in his "summary"

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of Bortkiewicz's solution: "(1) Write down the value relations; (2) take an eraser and rub them out; (3) finally write down the price relations—thus completing the so-called transformation process. The present elucidation should not rob Marx of esteem in the eyes of those who believe a subsistence wage provides valuable insights into the dynamic laws of motion of capitalism."

In this short but no less brilliant piece of analysis, Samuelson shows that given technology and the wage rate, an *equilibrium rate of profit* can be determined on the sole basis of supply-demand relations. Contrariwise, the argument developed in this paper is confined to the reverse case: given technology and the rate of profit, an *equilibrium wage rate* can be determined on the sole basis of supply-demand relations. It follows from this argument that what value relations really give us is just a way to determine the rate of profit by constraining supply-demand relations. On this basis, as will be shown in this paper, the transformation problem can be solved in the sense of transforming both input and output values while maintaining Marx's aggregate equalities.[1]

The formalization proposed in the paper is aimed at replicating Marx's key numerical examples in chapter 21 of volume II of *Capital*. These examples were the object of analysis and debate from the outset[2]; however, no comprehensive mathematical treatment of these examples appeared until the illuminating work by Donald Harris (1972) provided, for the first time, a formal analysis of Marx's inter-temporal equilibrium framework. On examining these examples, it has been useful to make the temporal dimension of the variables explicit by using superscript t as a given period, $t-1$ as the previous period, and so forth.[3]

In the following sections, Marx's model is built by first assuming that commodities are exchanged at their *actual values*, or labor-values expressed in monetary units. The formation of a general rate of profit is then considered by incorporating the subsequent prices of production into the equilibrium framework underlying the model, which is constrained in turn by imposing

[1]The conceptual framework outlined by Duncan Foley (1982) may be somewhat used to avoid Samuelson's criticisms, because in such a framework the *value* of labor-power is not necessarily given by any pre-determined bundle of subsistence goods. According to Foley (*ibid*.: 43), this traditional interpretation short-circuits the relation between the value of labor-power and the "value of money," and thus "makes money disappear as a mediating element in the situation." In the context of the transformation problem, this interpretation leads to the assumption that both nominal wage rate and rate of surplus-value—defined as the ratio of a division of value added between capitalists and workers—remain invariant in the transformation process, an approach first outlined by David Laibman (1973: 420-21). See also Lipietz (1982) and Duménil (1983-84). Instead of erasing labor-market relations in the formation of the *price* of labor-power, as the traditional point of view in fact does, the new interpretation indeed involves labor-market relations, although the nominal wage rate—in strong contrast with any other commodity price—remains unchanged in the transformation process.

[2]These examples were first explored, in different directions, by M. Tugan-Baranowsky, R. Luxemburg, N. Bukharin, O. Bauer, and H. Grossmann, among other authors. Sweezy (1970: ch. XI) provides an appraisal of these and other early commentators of Marx's theory of accumulation. For a more detailed discussion, see Rosdolsky (1977: ch. 30) and Mandel (1992).

[3]Although the system of equations proposed in this paper has its roots in Marx's own analysis of expanded reproduction, it shares an important aspect of the "iterative approach" to the transformation problem, first outlined by Bródy (1974) and independently explored by Morishima (1977) and Shaikh (1977); namely, that the transformation method does not necessarily prevent the differentiation over time between input prices and output prices. It should be noted that any algorithm of the (convergent) iterative method yields mere *numerical approximations* to the pre-determined solution of a system of simultaneous equations. However, since these algorithms are implemented by mean of a system of finite-difference equations, the real meaning of the sequence of "intermediate outcomes" generated by the system remains largely unclear in the iterative approach. In contrast, the system of equations proposed in this paper is not constrained by any pre-determined simultaneous solution; moreover, as we will see later in detail, the entire sequence of economic outcomes generated by the system can be explained in terms of an explicit process of capital transfers.

the two compensation conditions proposed in chapter 9 of volume III of *Capital*; namely, the sum of prices is equal to the sum of values, and the sum of profits is equal to the sum of surplus-value. The sequential solution obtained in this way is finally compared with those previously proposed by Bortkiewicz (1952, 1984), who derives prices from values in the special case of simple reproduction by making use of a set of assumptions that greatly differ from those adopted by Marx.[4]

The paper assumes that the scheme of expanded reproduction constitutes a natural way to deal with the quantitative analysis of capital accumulation developed by Marx. This approach was first proposed by Henryk Grossmann (1992), and is discussed in detail by Lapides (1982) and Kuhn (2007: ch. 5). Of course, there are various interpretations of Marx's text. In particular, Fred Moseley (1997: 182) argues that "the main purpose of Marx's reproduction tables was to refute Smith's dogma, the erroneous view that the total price of the total social product is entirely resolved into revenue." See also Freeman (1984: 262). However, the detailed discussion provided by Professor Moseley only tells us that the scheme can be used (and was really used by Marx) to assess Adam Smith's doctrine, among many other things, but this fact does not necessarily mean that it is the "most significant" feature of the scheme. Fortunately, the analysis of the mathematical structure of the scheme can be addressed quite apart from the issue of Marx's original intentions.[5]

2. Marx's Two-sector Scheme of Expanded Reproduction

This scheme refers to a private, closed economy in which credit-money and fixed capital are initially excluded from the account. A primary classification of commodities according to their use would rest on the distinction between two sectors or departments: commodities annually produced in order to produce other commodities (means of production) are allocated in department I; while those intended to directly satisfy individual needs (means of consumption) are allocated in department II. All the relevant magnitudes are expressed in terms of monetary value aggregates, and as such can provide only the conditions for aggregate equilibrium; hence, physical quantities are merely implicit, and do not form part of the defining problem. On the other hand, economic relations between industries can be formalized in terms of a system of finite-difference equations.

For the *j*th department ($j = 1, 2$), the nominal supply of commodities in the current period (D_j^t) is broken down by Marx into three parts,

$$D_j^t = C_j^{t-1} + V_j^{t-1} + S_j^t$$

where C_j^{t-1} represents the amount of money necessary to replace the means of production used up in the current period of production, or advanced *constant capital* (non-labor cost); V_j^{t-1} represents the amount of money necessary to renew the labor-power employed, or advanced *variable capital* (labor cost); and S_j^t represents the current expected profit, calculated as the difference between the value of supplied commodities and their total costs, or *surplus-value*. The time superscripts

[4]For Bortkiewicz, the method of transformation proposed by Marx is not only largely unjustified, but also utterly incapable of providing a coherent calculation of prices of production and the rate of profit. Moreover, Marx's schemes are seen by Bortkiewicz as a merely simplified way to deal with the "physical-surplus" analysis, first outlined by Dimitriev (1974) and later developed by Seton (1957), Sraffa (1963), and Morishima and Seton (1961), among other authors.

[5]Due to the lack of space, no formal solution is provided here of the systems of equations presented in the paper. However, for the reader interested in the mathematical details of these models, we have created several files with the program *Wolfram Mathematica*, which are at the disposal of the reader upon request to the authors.

are intended to capture the assumption that both constant and variable capital are *pre-determined* variables, whose values are therefore known *before* the current period of production takes place, while the surplus-value is only known *after* production itself has been effectively created.

According to Marx, commodities are supposed to exchange at their *actual values* when surplus-value is proportional to advanced variable capital; that is,

$$S_j^t = eV_j^{t-1}$$

where the parameter *e* is the *rate of surplus-value*.[6] This rate indexes the "degree of exploitation" of the social class of wage earners, and is given by the ratio of "surplus-labor" (unpaid labor) to "necessary-labor" (paid labor), which essentially arises from the process of production (*cf.* Marx 1976: 320-329).

2.1. Dynamic general equilibrium

Due to the parametric nature of the rate of surplus-value, it is possible to re-write sectoral supplies of commodities in terms of a given "mark up" over their labor costs; that is,

$$D_1^t = C_1^{t-1} + V_1^{t-1}(1 + e) \text{ (department I)} \quad (1)$$

$$D_2^t = C_2^{t-1} + V_2^{t-1}(1 + e) \text{ (department II)} \quad (2)$$

Equilibrium in the market of means of production is reached when the supply given by equation (1) is equal to the current demand of department I's output, which is given in turn by the total requirements of capital-goods from all industries in the following period of production, $C^t \equiv C_1^t + C_2^t$. These outlays involve both the demand required to maintain the scale of production unchanged, C^{t-1}, and that which is intended to expand this scale, or *constant capital accumulation*, that is $g_c^t\, C^{t-1}$, where g_c^t is the growth rate of aggregate constant capital. Hence,

$$D_1^t = C^t \text{ or } V_1^{t-1}(1 + e) = C_2^{t-1} + g_c^t C^{t-1} \quad (3)$$

The second expression in equation (3) can be derived as follows: $D_1^t = C^t \Rightarrow C_1^{t-1} + V_1^{t-1}(1+e) = C^{t-1} + g_c^t\, C^{t-1} \Rightarrow C_1^{t-1} + V_1^{t-1}(1+e) = C_1^{t-1} + C_2^{t-1} + g_c^t\, C^{t-1}$, and by erasing C_1^{t-1} in the last expression the above equation is easily obtained. This equilibrium equation (see Sweezy 1970: 176-77; Harris 1972: equation (14), p. 512) entails *simple reproduction* as a special case. In effect, if capital growth rates are set to zero, then equation (3) reduces to $V_1^{t-1} + V_1^{t-1} e = C_2^{t-1}$, for which net output (value added) of department I should be equal to department II's constant capital *in any period* (*cf.* Marx 1992: 478).

The current nominal demand for department II's output is given, on the other hand, by the part of the aggregate income that is eventually assigned by both workers and capitalists to their

[6]"Let us assume to start with that all commodities in the various spheres of production were sold at their actual values. What would happen then? According to our above arguments very different rates of profit would prevail in the various spheres of production.… If capitals that set in motion unequal quantities of living labour produce unequal amounts of surplus-value, this assumes that the level of exploitation of labour, or the rate of surplus-value, is the same, at least to a certain extent, or that the distinctions that exist here are balanced out by real or imaginary (conventional) grounds of compensation. This assumes competition among the workers, and an equalization that takes place by their constant migration between one sphere of production and another" (Marx 1991: 275).

personal consumption, that is $V^t + (1 - \alpha^t) S^t$, where α^t is the *average saving rate*. In this way, it is assumed that the workers do not save at all, since their demand for consumption is given by the total (pre-paid) nominal wages, $V^t \equiv V_1^t + V_2^t$. These outlays are derived from the overall demand for labor-power by the capitalists, which is, in turn, comprised of the demand required to maintain the scale of production unchanged, V^{t-1}, and the demand intended to expand this scale, or *variable capital accumulation*, that is $g_v^t V^{t-1}$, where g_v^t is the growth rate of aggregate variable capital. Since variable capital is here an index of employment level, this rate also measures employment growth; however, it must be supposed that the supply of labor-power is sufficient to satisfy a growing demand (*cf.* Marx 1992: 577). The demand for consumption by the capitalists is given, on the other hand, by the fraction of aggregate surplus-value ($S^t \equiv S_1^t + S_2^t$) that is not saved, $(1 - \alpha^t) S^t$. Thus, equilibrium in the market of means of consumption is given by

$$D_2^t = V^t + \left(1 - \alpha^t\right) S^t \text{ or } C_2^{t-1} + V_2^{t-1}\left(1 + e\right) = V^{t-1} + g_v^t V^{t-1} + \left(1 - \alpha^t\right) V^{t-1} e \qquad (4a)$$

Although equation (4a) can be perfectly used in formalizing Marx's model, it is possible to use instead the saving function which is implicit in equilibrium equations (3) and (4a). In effect, taking into account that $V^t = V_1^t + V_2^t$ by definition, and plugging equation (3) in (4a), yields

$$\alpha^t V^{t-1} e = g_c^t C^{t-1} + g_v^t V^{t-1} \qquad (4b)$$

This equation imposes that *aggregate saving* ($\alpha^t V^{t-1} e$) is equal to *aggregate investment*, given in turn by the sum of accumulated constant capital ($g_c^t C^{t-1}$) and accumulated variable capital ($g_v^t V^{t-1}$). The fact that *Say's law* is satisfied in Marx's scheme is not therefore an assumption by itself, but rather a consequence of the assumption of general equilibrium (*cf.* Should 1957: 616).

2.2. Restrictions on sectoral savings

Formally speaking, it is possible to treat the set of equations (1) to (4) as a system of finite-difference equations in which predetermined variables (C_1^{t-1}, C_2^{t-1}, V_1^{t-1}, and V_2^{t-1}) enter as *data* with respect to the current period, while current variables (D_1^t, D_2^t, g_c^t, g_v^t, and α^t) are treated as *unknowns*. On this basis, it would be possible to solve such a system recursively by starting from given initial values of predetermined variables and known values of parameters, if certain suitable restrictions on the capitalists' saving schedule were imposed.

The assumption that commodities exchange at their actual values presupposes that the surplus-value generated in each department cannot be invested or accumulated in any other department. This fact has two basic consequences on the specification of Marx's model: (i) the capitalists' saving must be equal to the amount of capital accumulated in the interior of each department; and (ii) the rate of profit may be unequal between departments. In this section, we deal with the first of these two consequences, leaving the analysis of differentiation in profitability to be addressed in the subsequent section.

The imposition of Say's law to departments I and II can be expressed by means of the following equations,

$$\alpha_1^t V_1^{t-1} e = g_{c1}^t C_1^{t-1} + g_{v1}^t V_1^{t-1} \qquad (5)$$

$$\alpha_2^t V_2^{t-1} e = g_{c2}^t C_2^{t-1} + g_{v2}^t V_2^{t-1} \qquad (6)$$

See Harris (1973: equations (9) and (10), p. 511). Here α_1^t and α_2^t stand for sectoral saving rates, g_{c1}^t and g_{c2}^t are sectoral growth rates of constant capital, and finally g_{v1}^t and g_{v2}^t are sectoral growth rates of variable capital. In all cases, numerical subscripts (*1, 2*) refer to departments I and II, respectively. On the other hand, both accumulated constant capital and accumulated variable capital for the economy seen as a whole can be expressed in terms of their corresponding sectoral rates of accumulation; hence,

$$g_c{}^t C^{t-1} = g_{c1}{}^t C_1^{t-1} + g_{c2}{}^t C_2^{t-1} \tag{7}$$

$$g_v{}^t V^{t-1} = g_{v1}{}^t V_1^{t-1} + g_{v2}{}^t V_2^{t-1} \tag{8}$$

These two equations (see Harris 1972: equations (11) and (12), p. 511) are necessary in order to link sectoral saving rates with the overall saving rate, α^t, which thereby becomes a weighted average of α_1^t and α_2^t. The augmented system of equations (1) to (8) now has *eleven* unknowns: the aforementioned five unknowns plus six new current variables: α_1^t, α_2^t, g_{c1}^t, g_{c2}^t, g_{v1}^t, and g_{v2}^t. Therefore, it is necessary to incorporate three additional independent equations to close the system. One of these equations can be provided by imposing additional restrictions on the capitalists' saving schedule; in this regard, Marx's own proposal consists simply of parameterizing the saving rate of department I, that is,

$$\alpha_1{}^t = \alpha_1 \tag{9}$$

where α_1 is a mere parameter. This restriction means that the capitalists' saving schedule is *exogenous* and, therefore, non-sensitive to changes in profit rates. An alternative restriction would be to assume instead a *uniform* saving rate across industries, that is, $\alpha_1^t = \alpha_2^t$, thereby rendering sectoral savings sensitive to changes in profit rates without essentially altering the mathematical structure of Marx's model.

2.3. Profitability and composition of capital

For the *j*th department, the *organic composition of capital*, which indexes the level of the productivity of labor, is defined as the ratio of constant to variable capital, $\pi_j^{t-1} = (C_j^{t-1}/V_j^{t-1})$, while the rate of profit is defined as the ratio of surplus-value to total advanced capital, $r_j^t = S_j^t / (C_j^{t-1} + V_j^{t-1})$. It follows from these definitions that sectoral rates of profit can be expressed solely in terms of sectoral organic compositions and the rate of surplus-value, that is,

$$r_1^t = V_1^{t-1} e / \left(C_1^{t-1} + V_1^{t-1}\right) = e / (1 + \pi_1{}^{t-1}) \tag{10}$$

$$r_2^t = V_2^{t-1} e / \left(C_2^{t-1} + V_2^{t-1}\right) = e / (1 + \pi_2{}^{t-1}) \tag{11}$$

Consequently, the rate of return on aggregate advanced capital, called by Marx the *average rate of profit*, is given by

$$\begin{aligned} r_a^t &= r_1^t \left(C_1^{t-1} + V_1^{t-1}\right) / \left(C^{t-1} + V^{t-1}\right) + r_2^t \left(C_2^{t-1} + V_2^{t-1}\right) / \left(C^{t-1} + V^{t-1}\right) \\ &= V^{t-1} e / \left(C^{t-1} + V^{t-1}\right) = e / (1 + \pi^{t-1}) \end{aligned} \tag{12}$$

where $\pi^{t-1} = C^{t-1}/V^{t-1}$ is the organic composition of aggregate capital. In equation (12), $r_a^{\,t}$ is expressed as a weighted average of sectoral profit rates, weights being the relative sizes of sectoral capitals on aggregate capital. In this way, *all* industries contribute *equivalently* to the formation of the average rate of profit. Seen as a whole, $r_a^{\,t}$ depends only on the composition of aggregate capital and the rate of surplus-value. Thus, if the productivity of labor of a given department is greater than that of the whole economy, $\pi_j^{t-1} > \pi^{t-1}$, then the rate of profit of such a department is forcibly smaller than the average, $r_j^{\,t} < r_a^{\,t}$, and *vice versa*; that is, if $\pi_j^{t-1} < \pi^{t-1}$ then $r_j^{\,t} > r_a^{\,t}$. Finally, for any department exhibiting the composition of aggregate capital, $\pi_j^{t-1} = \pi^{t-1}$, the profit rate should be equal to the average, $r_j^{\,t} = r_a^{\,t}$.

The system of equations (1) to (9) can now be extended by incorporating profit rates $r_1^{\,t}$, $r_2^{\,t}$, and $r_a^{\,t}$. However, these *current* variables are unknowns in equations (10), (11), and (12), respectively. Therefore, the augmented system of equations (1) to (12) must be completed by adding two independent equations without holding additional unknowns. Marx's basic proposal in this context is to parameterize the growth path of sectoral organic compositions, which depends mainly on the allocation of resources made by the capitalists of each department; hence,

$$a_1 g_{v1}^{\,t} = g_{c1}^{\,t} \tag{13}$$

$$a_2 g_{v2}^{\,t} = g_{c2}^{\,t} \tag{14}$$

Here, parameters a_1 and a_2 determine the type of technical change implemented in the two departments; for this reason we call them (sectoral) *coefficients of technical change*. If the organic composition of the *j*th department suffers an increase ($a_j > 1$) or a decrease ($a_j < 1$), then there is, respectively, labor-saving or capital-saving technical change. Finally, the constancy of the organic composition ($a_j = 1$) may be interpreted as the absence of technical change. Although technical change is exogenous in Marx's model, the coefficient of technical change of the economy (a^t) must be treated endogenously; thus,

$$a^t g_v^{\,t} = g_c^{\,t} \tag{15}$$

The coefficient a^t depends not only on the growth of sectoral organic compositions π_1^{t-1} and π_2^{t-1}, but also on the level of saving rate of department I given by equation (9); hence, a^t may eventually differ from unity even in the case in which a_1 and a_2 are set to 1. With the incorporation of this coefficient, the system of actual values is finally completed, since it now has fifteen unknowns and fifteen independent equations.

3. Exchange of Commodities at Their Prices of Production

The dynamic system formed by equations (1) to (15) rests on a *crucial* restriction, since the exchange at actual values forcibly prevents any transfers of surplus-value between sectors even though they may yield different rates of profit. To the contrary, the profit-maximizing behavior of firms' owners, in the absence of decisive obstacles to free competition between industries, will tend to eliminate any sustained difference between sectoral profit rates and the average rate of profit or, in other words, it will yield a *general* rate of profit (*cf.* Marx 1991: 252).

At this point, it is necessary to consider a few new concepts. To begin with, the *cost-price* of any set of commodities produced in a certain period of time (say, the natural year) is given by the sum of prices of the labor-power employed and the capital-goods used up in their production. In general, the cost-price will not be equal to the capital advanced in the same period of time, due to eventual differences in the *time of turnover* of capital. The clearest example is given by *fixed* *capital*, whose turnover time is typically greater than the annual period of production, and therefore the cost-price of the commodities produced with fixed capital involves only the *consumption of fixed capital* instead of its entire price.

However, Marx's scheme is built under the assumption that turnover time of capital is the same in all industries and is equal to the period of reference, which excludes, by definition, any outlays on fixed capital. Thus, in this purely circulating-capital model, the *j*th cost-price in terms of actual values is given by

$$K_j^{t-1} = \left(C_j^{t-1} + V_j^{t-1}\right)$$

As a consequence, the actual value of the *j*th department's output can be expressed in terms of a certain "mark-up" over its total cost-price; that is,

$$D_j^t = K_j^{t-1}\left(1 + r_j^t\right)$$

The fact that transfers of surplus-value between departments tend to cancel out any sustained difference in sectoral profit rates leads directly to the notion of *price of production*, which can be defined as the price of the commodity that, whatever the composition of capital advanced for its production may be, yields the general rate of profit, r^t (*cf*. Marx 1991: 257). It follows from this definition that the price of production of the *j*th department's output is given by

$$P_j^t = K_{pj}^{t-1}\left(1 + r^t\right)$$

where K_{pj}^{t-1} represents the *j*th cost-price when commodities are sold at their prices of production instead of their actual values. Consequently, there are two potential sources of divergence between sectoral prices (P_j^t) and sectoral values (D_j^t). First, sectoral *average profit,* $G_j^t = r^t K_{pj}^{t-1}$, can differ from surplus-value, $S_j^t = r_j^t K_j^{t-1}$, because sectoral profit rates can differ from the general rate of profit, that is $r_j^t \neq r^t$. Second, cost-prices can differ on their own, $K_{pj}^{t-1} \neq K_j^{t-1}$, since production prices can differ from actual values *in any period* (*cf*. Marx 1991: 308-9). Note that cost-prices K_{pj}^{t-1} and K_j^{t-1} are only divergent due to differences in price of the *same* amounts of input, which are assumed to be given; hence,

$$K_{pj}^{t-1} = m_c^{t-1} C_j^{t-1} + m_v^{t-1} V_j^{t-1}$$

where m_c^{t-1} and m_v^{t-1} are *indices of* (*constant and variable*) *cost-price deviation*. These indices measure the variation in price of capital-goods and labor-power between the two regimes, *i.e.* the variation in the production price of each type of input with respect to its actual value. In this way, if the production price of any such input is greater (smaller) than its value, then the corresponding cost-price index is greater (smaller) than unity, while if $m_c^{t-1} = 1 = m_v^{t-1}$, then $K_{pj}^{t-1} = K_j^{t-1}$.

3.1. General equilibrium under prices of production

On this basis, it is not difficult to formalize Marx's two-sector scheme of expanded reproduction under the assumption that commodities exchange at their prices of production. To begin with, departments' supplies of commodities evaluated at their prices of production are given by

$$P_1^t = \left(m_c^{t-1} C_1^{t-1} + m_v^{t-1} V_1^{t-1}\right)\left(1 + r^t\right) \text{ (department I)} \tag{16}$$

$$P_2^t = \left(m_c^{t-1}C_2^{t-1} + m_v^{t-1}V_2^{t-1}\right)\left(1 + r^t\right) \text{ (department II)} \tag{17}$$

Here m_c^{t-1} and m_v^{t-1} index the deviation in *input* prices, while the deviation in *output* prices is indexed instead by (P_1^t/D_1^t) and (P_2^t/D_2^t) for departments I and II, respectively. As the notation makes clear, input prices refer to period *t-1* and output prices to period *t* in the two regimes. As a consequence, it is perfectly possible that $m_c^{t-1} \neq (P_1^t/D_1^t)$. On the other hand, it must be borne in mind that the price of the labor input is not necessarily the same as the output price of department II or any given sub-division of this department; therefore, in general, not only $m_v^{t-1} \neq (P_2^t/D_2^t)$, but also $m_v^t \neq (P_2^t/D_2^t)$. It follows from this formalization that both nominal and real wages are allowed to vary in the transformation process.

The transformation method proposed in this paper rests on a *price adjustment* which does not affect the framework of inter-temporal general equilibrium in any way. Since physical quantities remain unchanged in the transformation, a new equilibrium in the capital-goods market is reached when the current supply given by (16) is equal to the corresponding current demand, given by the aggregate capital advanced to acquire the means of production that are to be used up in the *subsequent* period of production evaluated at prices of production, $m_c^t C^t$; hence,

$$P_1^t = m_c^t C^t \quad \text{or} \quad \left(m_c^{t-1}C_1^{t-1} + m_v^{t-1}V_1^{t-1}\right)\left(1 + r^t\right) = m_c^t C^t \tag{18}$$

In this equilibrium equation, m_c^{t-1} indexes the variation in price of the material input used up in the current period of production, while m_c^t indexes the variation in price of department I's output, which enters only as material input of all departments in the *subsequent* period of production. In effect, since under equilibrium $D_1^t = C^t$, then $m_c^t = (P_1^t/D_1^t)$ and forcibly $m_c^{t-1} = (P_1^{t-1}/D_1^{t-1})$; however, these two indices can differ over time in the course of reproduction, as shown below by way of example.

Under equilibrium in the market of means of consumption, the supply given by (17) must be equal to the corresponding demand, given by the workers' total income plus the part of the capitalists' income which is intended for consumption, $m_v^t V^t + (1-\beta^t)\, G^t$; hence,

$$P_2^t = m_v^t V^t + \left(1-\beta^t\right)G^t \text{ or}$$

$$\left(m_c^{t-1}C_2^{t-1} + m_v^{t-1}V_2^{t-1}\right)\left(1 + r^t\right) = m_v^t V^t + \left(1-\beta^t\right)\left(m_c^{t-1}C^{t-1} + m_v^{t-1}V^{t-1}\right)r^t \tag{19a}$$

Here β^t represents the *average rate of investment*, and $G^t \equiv G_1^t + G_2^t$ is the *aggregate average profit*, calculated by applying the general rate of profit over aggregate advanced capital, $G^t = (m_c^{t-1}C^{t-1} + m_v^{t-1}V^{t-1})\, r^t$. The workers' current demand for consumption, on the other hand, is given by their aggregate (pre-paid) nominal wages, $m_v^t V^t$, which may eventually involve the accumulation of variable capital and the subsequent expansion of the employment level. Plugging equation (18) in (19a), and after minor manipulations and rearrangements, we obtain

$$\beta^t\left(m_c^{t-1}C^{t-1} + m_v^{t-1}V^{t-1}\right)r^t = \left(m_c^t C^t - m_c^{t-1}C^{t-1}\right) + \left(m_v^t V^t - m_v^{t-1}V^{t-1}\right) \tag{19b}$$

Aggregate investment, in the left-hand side of (19b), must be equal in any period to the sum of accumulated constant capital and accumulated variable capital, in the right-hand side of (19b). General equilibrium is thus satisfied if and only if equations (18) and (19a)—or, equivalently, equations (18) and (19b)—simultaneously hold. On this basis, *sectoral rates of investment* (β_1^t and β_2^t) can be incorporated into the model as follows,

$$\beta_1^{t}\left(m_c^{t-1}C_1^{t-1} + m_v^{t-1}V_1^{t-1}\right)r^{t} = \left(m_c^{t}C_1^{t} - m_c^{t-1}C_1^{t-1}\right) + \left(m_v^{t}V_1^{t} - m_v^{t-1}V_1^{t-1}\right) \tag{20}$$

$$\beta_2^{t}\left(m_c^{t-1}C_2^{t-1} + m_v^{t-1}V_2^{t-1}\right)r^{t} = \left(m_c^{t}C_2^{t} - m_c^{t-1}C_2^{t-1}\right) + \left(m_v^{t}V_2^{t} - m_v^{t-1}V_2^{t-1}\right) \tag{21}$$

Here β_1^t and β_2^t measure only the relative size of capital accumulation made by the capitalists as a whole *within* each department, because the surplus-value generated and saved in a given industry can now be freely invested in any other industry, in such a way as to ensure that every investor merely obtains the average profit. It follows from this rule that sectoral saving rates will differ in general from sectoral rates of investment (*cf.* Harris 1972: 513ff).

3.2. Marx's laws of compensation

The system of equations (16) to (21) now entails eight *unknowns* or current variables (P_1^t, P_2^t, m_c^t, m_v^t, r^t, β^t, β_1^t, and β_2^t), whereas there are only six equations. It is therefore necessary to incorporate two additional independent equations to complete the system of prices of production. Since a pure price adjustment does not change the physical quantities actually involved in the scheme,[7] a direct way to provide such equations is simply to impose the two famous "invariance postulates" or, as Marx called them "laws of compensation." First, the sum of sectoral prices must be equal to the sum of sectoral values

$$P_1^{t} + P_2^{t} = D_1^{t} + D_2^{t} \quad \text{or}$$

$$\left(m_c^{t-1}C^{t-1} + m_v^{t-1}V^{t-1}\right)\left(1 + r^{t}\right) = \left(C^{t-1} + V^{t-1}\right)\left(1 + r_a^{t}\right) \tag{22a}$$

And second, the sum of sectoral (average) profits must be equal to the sum of sectoral surplus-value,

$$G_1^{t} + G_2^{t} = S_1^{t} + S_2^{t} \quad \text{or} \quad \left(m_c^{t-1}C^{t-1} + m_v^{t-1}V^{t-1}\right)r^{t} = \left(C^{t-1} + V^{t-1}\right)r_a^{t} \tag{23a}$$

The first law of compensation ensures that the price level remains unchanged in the transformation, which is coherent with Marx's key assumptions about the money-commodity account (*cf.* Marx 1992: 400-401). The second law of compensation obliges aggregate surplus-value to remain unchanged in the transformation. It should be noted that equations (22a) and (23a) are satisfied if and only if the following two equations simultaneously hold,

$$\left(m_c^{t}C^{t} + m_v^{t}V^{t}\right) = \left(C^{t} + V^{t}\right) \tag{22b}$$

$$r^{t} = r_a^{t} \tag{23b}$$

[7]Although Marx assumes that products exchange at their values in the analysis of the scheme of reproduction, he stresses that "in as much as prices diverge from values, this circumstance cannot exert any influence on the movement of the social capital. The same mass of products is exchanged afterwards as before, even though the value relationships in which the individual capitalists are involved are no longer proportionate to their respective advances and to the quantities of surplus-value produced by each of them" (Marx 1992: 469).

The assumption that aggregate total capital remains unchanged in the transformation, given by equation (22b), is coherent with Marx's contention that the "movement" of social capital is unaffected by the formation of prices of production. Since the rate of profit, on the other hand, is given exogenously by equation (23b), it cannot be an "equilibrium" rate of profit, because it does not depend on equilibrium equations (18) and (19a). Taken together, equations (22b) and (23b) enable Say's law to hold in the economy as a whole; that is $\beta^t = \alpha^t$. Note that equations (22b) and (23b) are not additional "invariance postulates," since they are mere corollaries of equations (22a) and (23a).

Given that there are eight unknowns and eight independent equations in the system-of-production-prices (hereafter SPP), given by equations (16) to (23), this system can be solved recursively by beginning from the solution of the system-of-actual-values (hereafter SAV), given by equations (1) to (15), and from the known initial values of the indices of cost-price deviation. It should be noted that the SPP solution does not alter the path of capital accumulation and economic growth given by the SAV solution, as shown in the following section.

4. Some Numerical Examples

It is possible to evaluate the system of equations (1) to (15), or SAV, at the initial dataset used by Marx in his first numerical example in chapter 21 of volume II of *Capital*, that is, $C_1^0 = 4000$, $V_1^0 = 1000$, $C_2^0 = 1500$, $V_2^0 = 750$, and $e = 1$, $a_1 = 1$, $a_2 = 1$, and $\alpha_1 = 0.5$. The numerical results, for the first three years of the series only, derived from the SAV recursive solution, are summarized in Table 1, which exactly replicates the results originally obtained by Marx.

For the reader interested in Marx's own comments on this example (*cf.* Marx 1992: 586-589), it must be borne in mind that he sometimes breaks down the process of capital accumulation into intermediate steps.[8] Note that the capitalists' saving schedule does not appear to follow any explicit rule here, while the saving rate of department II suffers a strongly asymmetric response to capital growth in year 2. Several commentators have complained about this apparent anomaly (*cf.* Robinson 1984: 19; Morishima 1977: 118-122). However, sectoral saving rates in Marx's model *indeed do* satisfy an equilibrium condition, given by

$$\left(\alpha_2^{\,t} / \alpha_1^{\,t}\right) = \left(1 + \pi_2^{\,t-1}\right) / \left(1 + \pi_1^{\,t-1}\right)$$

See Harris (1972: equation (16), p. 512). Note that from t=2 onwards in Marx's example α_1^t is approximately 1.67 times α_2^t, which is due to the given distribution of the departments' organic compositions, $(1+\pi_2^{t-1})/(1+\pi_1^{t-1}) = 3/5$ (see Harris 1972: footnote 16, p. 512). It is worth noting that the above rule is not satisfied *in the first period* of this example due to Marx's own selection of department I's saving rate. However, it would be perfectly possible to maintain the structure of saving rates unchanged from the outset by imposing *Harris's rule* to the first period, or equivalently by choosing $\alpha_1 = 0.45$ instead of $\alpha_1 = 0.5$.

In order to simplify the subsequent exposition, a second example has been prepared that allows Harris's rule to hold in the first period as well. In this case, we start from a dataset used by Bortkiewicz (1984: Table 1, p. 204), namely, $C_1^0 = 225$, $V_1^0 = 90$, $C_2^0 = 150$, $V_2^0 = 210$. In order

[8]For instance, in year 2 the accumulation of constant and variable capital in department II is decomposed as follows: first, "if we initially leave aside the money here… then a further 25_v must be laid out for 50_c"; and second, since "means of consumption to the value of 110 are consumed by the workers in department I instead of by the capitalists of department II," the latter are "forced to capitalize this 110_s instead of consuming it…. But if department II transforms this 110 into additional constant capital, it needs a further additional variable capital of 55" (Marx 1992: 587-588). Note that by taking Marx's steps together, the figures of Table 1 are easily obtained, that is $g_{c2}^2\, C_2^1 = 50_c + 110_c = 160$, and $g_{v2}^2\, V_2^1 = 25_v + 55_v = 80$.

Table 1. Marx's Two-sector Scheme of Expanded Reproduction: Actual Values.

	Total Capital	Constant Capital	Variable Capital	Organic Compo-sition of Capital	Rate of Surplus Value	Surplus Value	Rate of Profit	Rate of Saving	Capitalists Consump-tion	Total Saving	Constant Capital Accumu-lation	Variable Capital Accumu-lation	Coefficient of Technical Change	Value of Product
Marx's example: Year 1														
Depart. I	5000	4000	1000	4	1	1000	0.2	0.5	500	500	400	100	1	6000
Depart. II	2250	1500	750	2	1	750	0.333	0.2	600	150	100	50	1	3000
Total	7250	5500	1750	3.142	1	1750	0.241	0.371	1100	650	500	150	1.060	9000
Marx's example: Year 2														
Depart. I	5500	4400	1100	4	1	1100	0.2	0.5	550	550	440	110	1	6600
Depart. II	2400	1600	800	2	1	800	0.333	0.3	560	240	160	80	1	3200
Total	7900	6000	1900	3.157	1	1900	0.24	0.415	1110	790	600	190	1	9800
Marx's example: Year 3														
Depart. I	6050	4840	1210	4	1	1210	0.2	0.5	605	605	484	121	1	7260
Depart. II	2640	1760	880	2	1	880	0.333	0.3	616	264	176	88	1	3520
Total	8690	6600	2090	3.157	1	2090	0.24	0.415	1221	869	660	209	1	10780
Bortkiewicz's example: Year 1														
Depart. I	315	225	90	2.5	1	90	0.286	0.28	64.8	25.2	18	7.2	1	405
Depart. II	360	150	210	0.714	1	210	0.583	0.137	181.2	28.8	12	16.8	1	570
Total	675	375	300	1.25	1	300	0.444	0.18	246	54	30	24	1	975
Bortkiewicz's example: Year 2														
Depart. I	340.2	243	97.2	2.5	1	97.2	0.286	0.28	70	27.2	19.4	7.8	1	437.4
Depart. II	388.8	162	226.8	0.714	1	226.8	0.583	0.137	195.7	31.1	13.0	18.1	1	615.6
Total	729	405	324	1.25	1	324	0.444	0.18	265.7	58.3	32.4	25.9	1	1053
Bortkiewicz's example: Year 15														
Depart. I	925.2	660.9	264.3	2.5	1	264.3	0.286	0.28	190.3	74	28.0	11.2	1	1189.6
Depart. II	1057.4	440.6	616.8	0.714	1	616.8	0.583	0.137	532.2	84.6	19.7	27.5	1	1674.2
Total	1982.6	1101.4	881.2	1.25	1	881.2	0.444	0.18	722.5	158.6	47.7	38.7	1	2863.8

to deal with *balanced expanded reproduction*, we select $e = 1$, $a_1 = 1$, $a_2 = 1$, and $\alpha_1 = 0.28$. Table 1 also summarizes the SAV solution of Bortkiewicz's example for years 1, 2, and 15.

The main result of this second example deals with an annual growth rate of 8 percent for capital, labor, and output, in each department and in the economy as a whole. It must be borne in mind that this rate also measures the *real* growth of the system; in fact, SAV's dynamic behavior can be seen as the real counterpart of SPP's dynamic behavior, since unitary values (actual values per unit of output) are supposed to remain unchanged over time under the assumption of constant organic composition.

At this point, all that is needed to evaluate the system of equations (16) to (23), or SPP, by beginning from Bortkiewicz's example in Table 1, is to set the initial values of the indices of cost-price deviation, m_c^0 and m_v^0, under the sole condition that equation (22b) must be also satisfied in period 0. A first, natural choice is that proposed by Marx; namely, $m_c^0 = 1$ and $m_v^0 = 1$, which imposes no deviation in cost-prices for the first period (*partially transformed cost-prices*). A second, crucial choice is to set $m_c^0 = 1.206$ and $m_v^0 = 0.742$, which brings about full deviation in cost-prices from the first period (*fully transformed cost-prices*). The main figures of the SPP solution for years 1, 2, and 15 are summarized in Table 2.

By comparing any year of the series in Table 1 (Bortkiewicz's example) and in Table 2, the typical figures of Marx's original solution are attained: the redistribution of surplus-value between industries causes the price of department I's output to be higher than its value, and the price of department II's output to be lower than its value. This behavior arises due to the fact that the organic composition of capital in department I is greater than that of department II.

In year 1 (partially transformed cost-prices in Table 2), the constant capital necessary to maintain the scale of production unchanged is revaluated from $C^0 = 375$ to $m_c^1 C^0 = 421.3$ due to the increase in price of capital-goods, $m_c^1 = P_1^1/D_1^1 = 455/405 = 1.123$; an amount of money capital of 46.3 must therefore be tied up in department I for this purpose. This "tying-up" of capital is financed by the "release" of an equivalent portion of the formerly advanced variable capital, which has been devaluated from $V^0 = 300$ to $m_v^1 V^0 = 253.7$ due to the reduction in nominal wages, $m_v^1 = 0.846$. On the other hand, the actual accumulation of constant capital evaluated at current prices of production is $(C^1 – C^0)\, m_c^1 = 33.7$, and that of variable capital is $(V^1 – V^0)\, m_v^1 = 20.3$; as a consequence, aggregate (net) investment amounts to $\beta^1 G^1 = 33.7 + 20.3 = 54$. The twofold process of actual accumulation and the "tying-up" and "release" of capital explains that net investment for capital-goods amounts to $(C^1 – C^0)m_c^1 + C^0(m_c^1–1) = 33.7 + 46.3 = 80$, and to $(V^1 – V^0)m_v^1 + V^0(m_v^1–1) = 20.3 – 46.3 = –26$ for labor-power. The decrease in price experienced by department II's output, $P_2^1/D_2^1 = 520/570 = 0.912$, is thus consistent with the reduction of the demand for consumption by the workers, $m_v^1 V^1 – V^1 = 274 – 324 = –50$, since the capitalists' demand for consumption remains unchanged; that is, $G^1\,(1–\beta^1) = 246 = S^1\,(1–\alpha^1)$.[9]

[9]The *monetary* mechanism of "tying-up" and "release" of capital was extensively documented by Marx in chapter 6 of volume III of *Capital*: "By the tying-up of capital we mean that, out of the total value of the product, a certain additional proportion must be transformed back into the elements of constant or variable capital, if production is to continue on its old scale. By the release of capital we mean that a part of the product's total value which previously had to be transformed back into either constant or variable capital becomes superfluous for the continuation of production on the old scale and is now available for other purposes" (Marx 1991: 205). In fact, if wages fall, then there is a "release" of capital: "For capital that is newly invested, this has simply the effect of enabling it to function at an increased rate of surplus-value. The same quantity of labour is set in motion with less money than before, and in this way the unpaid portion of labour is increased at the cost of the paid portion. But for capital that was already invested earlier, not only does the rate of surplus-value increase, but on top of this a portion of the capital previously laid out on wages is set free. This was formerly tied up and formed a portion constantly deducted from the proceeds of production, a portion which was laid out on wages and had to function as variable capital if the business was to proceed on the old scale. This portion now becomes available and can be used for new capital investment, whether to extend the same business or to function in another sphere of production" (*ibid.*: 210).

Table 2. Marx's Two-sector Scheme of Expanded Reproduction: Prices of Production.*

	Total Capital	Constant Capital	Variable Capital	Index of Constant Cost-Price Deviation	Index of Variable Cost-Price Deviation	Value Compo-sition of Capital	Rate of Profit	Average Profit	Rate of Investment	Capitalists Consump-tion	Total Investment	Constant Capital Accumu-lation	Variable Capital Accumu-lation	Price of Product	Index of Price-Value Deviation
Year 1 (Partially transformed cost-prices)															
Dep. I	315	225	90	1	1	2.5	0.444	140	0.287	99.8	40.2	48	−7.8	455	1.123
Dep. II	360	150	210	1	1	0.714	0.444	160	0.086	146.2	13.8	32	−18.2	520	0.912
Total	675	375	300	1	1	1.25	0.444	300	0.18	246	54	80	−26	975	1
Year 2 (Partially transformed cost-prices)															
Dep. I	355.2	273	82.2	1.123	0.846	3.321	0.444	157.9	0.221	123	34.9	34.8	0.1	513.1	1.173
Dep. II	373.8	182	191.8	1.123	0.846	0.949	0.444	166.1	0.141	142.7	23.4	23.2	0.2	539.9	0.877
Total	729	455	274	1.123	0.846	1.661	0.444	324	0.18	265.7	58.3	58	0.3	1053	1
Year 15 (Fully transformed cost-prices)															
Dep. I	993.3	797.1	196.2	1.206	0.742	4.062	0.444	441.5	0.18	362	79.5	63.8	15.7	1434.8	1.206
Dep. II	989.3	531.4	457.8	1.206	0.742	1.161	0.444	439.7	0.18	360.5	79.1	42.5	36.6	1429	0.853
Total	1982.6	1328.5	654.1	1.206	0.742	2.031	0.444	881.2	0.18	722.5	158.6	106.3	52.3	2863.8	1
Year 1 (Fully transformed cost-prices)															
Dep. I	338.2	271.4	66.8	1.206	0.742	4.063	0.444	150.3	0.18	123.3	27.1	21.7	5.3	488.5	1.206
Dep. II	336.8	180.9	155.9	1.206	0.742	1.161	0.444	149.7	0.18	122.7	26.9	14.5	12.5	486.5	0.853
Total	675	452.3	222.7	1.206	0.742	2.031	0.444	300	0.18	246	54	36.2	17.8	975	1
Year 2 (Fully transformed cost-prices)															
Dep. I	365.3	293.1	72.1	1.206	0.742	4.063	0.444	162.3	0.18	133.1	29.2	23.4	5.8	527.6	1.206
Dep. II	363.7	195.4	168.3	1.206	0.742	1.161	0.444	161.7	0.18	132.6	29.1	15.6	13.5	525.4	0.853
Total	729	488.5	240.5	1.206	0.742	2.031	0.444	324	0.18	265.7	58.3	39.1	19.2	1053	1

(*) SPP's recursive solution by starting from SAV's solution of Bortkiewicz's example in Table 1

In year 2 (partially transformed cost-prices in Table 2), the process of revaluation and devaluation of capital is repeated. The constant capital necessary to maintain the scale of production unchanged is revaluated from $m_c^1C^1 = 455$ to $m_c^2C^1 = 475$ due to the increase in price of capital-goods, $m_c^2/m_c^1 = 1.173/1.123 = 1.044$; therefore, an amount of money-capital of 20 must be tied up in department I. This "tying-up" of capital is financed by the "release" of an equivalent portion of the formerly advanced variable capital, which is devaluated accordingly from $m_v^1V^1 = 274$ to $m_v^2V^1 = 254$ due to the reduction in nominal wages, $m_v^2/m_v^1 = 0.784/0.846 = 0.927$. On the other hand, the accumulation of constant capital evaluated at current prices of production is $(C^2 - C^1)\, m_c^2 = 38$, and that of variable capital is $(V^2 - V^1)\, m_v^2 = 20.3$; as a result, aggregate net investment amounts to $\beta^2G^2 = 38 + 20.3 = 58.3$. This result is consistent with the fact that net investment for capital-goods amounts to $(C^2 - C^1)\, m_c^2 + C^1\, (m_c^2 - m_c^1) = 38 + 20 = 58$, while net investment in labor-power amounts to $(V^2 - V^1)\, m_v^2 + V^1\, (m_v^2 - m_v^1) = 20.3 - 20 = 0.3$. Observe that aggregate net investment, $\beta^2G^2 = 58 + 0.3 = 58.3$, is distributed accordingly between the two departments; that is, $\beta_1^2G_1^2 = (C_1^2m_c^2 - C_1^1m_c^1) + (V_1^2\, m_v^2 - V_1^1m_v^1) = 34.8 + 0.1 = 34.9$ for department I, and $\beta_2^2G_2^2 = (C_2^2m_c^2 - C_2^1m_c^1) + (V_2^2m_v^2 - V_2^1m_v^1) = 23.2 + 0.2 = 23.4$ for department II. The mechanism of "tying-up" and "release" of capital ceases to operate in year 15, where price-value deviations also stop changing.

Moreover, changes in the prices of capital-goods and labor-power also lead to changes in the "value" (or nominal) composition of capital, given at any period by $(m_c^t/m_v^t)\, \pi_j^t$ for the jth department. These changes become smaller over time, however, and virtually disappear in year 15, since the proportionality factor converges at $(m_c^{15}/m_v^{15}) = 1.6$. Given that from this year onwards there is no variation in relative prices, then the figures of year 15 become in fact those of fully transformed cost-prices.

Note that sectoral investment rates tend to converge towards the average rate of investment in the SPP model. In fact, the *equilibrium condition* for the distribution of net investment between departments when commodities are exchanged at their *full prices of production* is given by

$$\beta_1^t = \beta_2^t$$

See Harris (1972: equation (16a), p. 517). However, the SPP recursive solution does not require this condition to hold when cost-prices are only partially transformed, due to the activation of the aforementioned mechanism of the "tying-up" and "release" of capital. Nevertheless, all that is needed to obtain full prices of production from the outset in the SPP model is to impose *Harris's rule* above or, what amounts to the same, to re-direct the SPP solution by setting the initial values of cost-price indices to their convergent values of year 15, that is $m_c^0 = 1.206$ and $m_v^0 = 0.742$, which leads us to the results reported in Table 2 (fully transformed cost-prices). In this case, the mechanism of the "tying-up" and "release" of capital is completely de-activated, and the *growth path* given by the SPP solution is then exactly the same as that given by the SAV solution. In fact, even by starting from arbitrary initial conditions, the SPP solution is convergent with the same set of full prices of production and, therefore, with the same dynamic behavior as that of the SAV solution.[10]

[10]In such a situation, the system indeed reaches *balanced expanded reproduction*. As Donald Harris (1972: 517) points out: "For growth to be balanced, it is then only necessary that the capitalists reinvest the same proportion of their profits. This is what (16a) says. The difference between the two conditions reflects the fact that one group of capitalists gains at the expense of the other due to the role of prices and competition." In this way, the *full equilibrium solution* derived by Professor Harris on the basis of his pioneering restatement of Marx's scheme of accumulation can be seen as the *convergent equilibrium solution* of the SAV-SPP model. On the other hand, the eventual formation of full prices of production has nothing to do with the fact that prices of production can be interpreted, in any period, as "centers of gravity" of observed market prices. This is nonetheless a question that lies clearly outside the scope of this paper, which is confined to conditions of competitive equilibrium.

Observe that the SPP full equilibrium solution yields, in any period, an increase in price of department I's output of $(P_1^t/D_1^t) = 1.206$, a decrease in price of department II's output of $(P_2^t/D_2^t) = 0.853$, and a decrease in price of labor-power of $m_v^t = 0.742$. Since the reduction in price of consumer goods (14.7 percent) is smaller than the reduction in nominal wage rate (25.8 percent), then the *real wage rate* suffers a decrease, which means that the transformation process entails real changes in income distribution between capitalists and workers. For example, when commodities are evaluated at their actual values, the workers are able to buy $(V^0 + g_{v1}{}^1V^0)/D_2{}^1 = (300 + 24)/570 = 56.8$ percent of the consumer goods available (see Table 1, Bortkiewicz's example, year 1), but they can only buy $(V^0m_v{}^1 + g_{v1}{}^1V^0m_v{}^1)/P_2{}^1 = (227.7 + 17.8)/486.5 = 50.5$ percent when these commodities are evaluated at their prices of production (see Table 2, year 1); as a result, the loss of purchasing power suffered by the workers amounts to 6.3 points. This re-distribution of income implies a higher rate of surplus-value, which alters the proportion between unpaid labor and paid labor given in terms of actual values.

Note that capital transfers between industries have the basic effect of changing the distribution of aggregate demand—including the demand for labor-power—between the different markets. In the example above, there is an increase in the demand for means of production and a decrease in the demand for consumption, which are concurrent results of the very same process. Under the pressure of maximizing the rate of profit, on the one hand, part of the demand previously laid out on wages—previously tied up within the market for labor—is set free and then redirected to the market of means of production; hence, the price of means of production tends to rise, while the price of labor-power tends to fall. On the other hand, the reduction in nominal wages translates into an equivalent reduction in workers' demand for consumption, which in turn causes the price of means of consumption to fall, since the demand for consumption by the capitalists remains unchanged. Finally, this reduction in price of consumer goods determines the equilibrium level of real wages; in particular, if the reduction in prices does not entirely compensate the reduction in nominal wages, as in the example above, then real wages should also fall.

This result raises the following question: is it consistent with Marx's theory to argue that profit rate equalization influences the real wage bundle? In this regard, it is worth noting first that we do not claim that this result is simply an expected, automatic result from Marx's theory. Certainly, the textual evidence only tells us that Marx clearly identifies the mechanism discussed above of re-distribution of demand between the different markets—including the market for labor—and its main consequences. However, it must be recognized that he did not analyze the formation of prices of production in such terms, since his treatment of the transformation problem (vol. III) does not rest on the scheme of reproduction (vol. II). Nevertheless, we are convinced that the integration of both instances proposed in this paper is not inconsistent with Marx's theory.

5. The Case of Simple Reproduction

At this point, it seems useful to deal with the special case of *simple reproduction*, which allows us to compare the SPP solution with some well-known solutions formerly proposed in the literature on this subject. In order to reach simple reproduction, the SAV solution is re-directed by setting $e = 2/3$, $a_1 = 1$, $a_2 = 1$, and $\alpha_1 = 0$ (Bortkiewicz's original example). Although it is perfectly possible to obtain SPP's recursive solution by starting again from $m_c{}^0 = 1$ and $m_v{}^0 = 1$, the analysis performed in this section is confined, for reasons of space, to the case of full prices of production, which can be directly obtained by solving the following system of simultaneous equations,

$$P_1 = \left(m_c C_1 + m_v V_1\right)\left(1 + r\right) \tag{24}$$

$$P_2 = \left(m_c C_2 + m_v V_2\right)\left(1 + r\right) \tag{25}$$

$$\left(m_c C_1 + m_v V_1\right)\left(1 + r\right) = m_c C \tag{26}$$

$$m_c C + m_v V = C + V \tag{27}$$

$$r = e\ V / \left(C + V\right) \tag{28}$$

This system is a special case of the SPP model under the assumption of simple reproduction, in which all variables involved are formally treated "as if" they were calculated "at the same time," thereby avoiding any explicit indication of their temporal dimension. In this case, equations (24) and (25) are the simultaneous versions of equations (16) and (17), while equilibrium equations (18) and (19) are reduced to equation (26). Recall that equations (27) and (28), simultaneous counterparts of equations (22b) and (23b), respectively, are mere corollaries of Marx's laws of compensation, given by

$$P_1 + P_2 = D_1 + D_2 \quad \text{or} \quad \left(m_c C + m_v V\right)\left(1 + r\right) = C + V\left(1 + e\right) \tag{29}$$

$$G_1 + G_2 = S_1 + S_2 \quad \text{or} \quad \left(m_c C + m_v V\right) r = V\ e \tag{30}$$

Given that constant capital $C \equiv C_1 + C_2$, variable capital $V \equiv V_1 + V_2$, and the rate of surplus-value e are treated as data, there are only five unknowns (P_1, P_2, m_c, m_v, and r) in the system of equations (24) to (28). Since r is given exogenously by (28), we are allowed to impose the rate of profit to the system's equilibrium, given by (26), so as to directly obtain m_c and m_v from equations (26) and (27); finally, r is subsequently employed to calculate P_1 and P_2 through equations (24) and (25). The numerical results derived from both the SAV and the SPP solutions are summarized in Table 3.

It must be borne in mind that the intermediate steps that connect SAV's outcomes with SPP's outcomes, which again involve the "tying-up" and "release" of money-capital and subsequent divergences between (zero) sectoral saving rates and (non-zero) sectoral rates of investment, have been omitted in Table 3.

5.1. Previous solutions of the transformation problem

Although the proponents of previous solutions of the transformation problem frequently deal with a three-sector model, it should be noted that these solutions are supposed to be applicable to any number of sectors or departments. It therefore does not seem unjustified to unfold these solutions in the two-sector model analyzed in this paper. In this context, we distinguish between three major solutions, which can be obtained by following Bortkiewicz (1952, 1984), although it is always possible to obtain other solutions along the same line, that is by substituting Marx's two laws of compensation with any other *ad hoc* pair of independent equations.

Solution 1. The strong reaction of Bortkiewicz (1952: 13-17) against equation (29), largely based on the contention that it does not fulfil the condition that the unit of price should be the same as the unit of value, leads him to identify this common unit of measurement in the sector producing the money-commodity (gold), which is ascribed to (the "luxury" subdivision of) department II. Thus Bortkiewicz substitutes equation (29) with the following equation

Table 3. Alternative Schemes of Simple Reproduction: Actual Values and Prices of Production.

	Total Capital	Constant Capital	Variable Capital	Index of Constant Cost-Price Deviation	Index of Variable Cost-Price Deviation	Composition of Capital	Rate of Surplus Value	Rate of Profit	Surplus Value/ Average Profit	Value Added	Price of Product	Index of Price-Value Deviation
Actual Values: SAV's solution												
Depart. I	315	225	90	-	-	2.5	0.667	0.19	60	150	375	-
Depart. II	360	150	210	-	-	0.714	0.667	0.389	140	350	500	-
Total	675	375	300	-	-	1.25	0.667	0.296	200	500	875	-
Prices of Production: SPP's solution												
Depart. I	331.4	257.7	73.6	1.145	0.818	3.5	1.333	0.296	98.2	171.8	429.5	1.145
Depart. II	343.6	171.8	171.8	1.145	0.818	1	0.593	0.296	101.8	273.6	445.5	0.891
Total	675	429.5	245.5	1.145	0.818	1.75	0.815	0.296	200	445.5	875	1
Prices of Production: Previous Solution 1												
Depart. I	369.8	279.8	90	1.243	1	3.108	1.072	0.261	96.5	186.5	466.3	1.243
Depart. II	396.5	186.5	210	1.243	1	0.888	0.493	0.261	103.5	313.5	500	1
Total	766.3	466.3	300	1.243	1	1.554	0.667	0.261	200	500	966.3	1.104
Prices of Production: Previous Solution 2												
Depart. I	297.4	225	72.4	1	0.804	3.108	1.072	0.261	77.6	150	375	1
Depart. II	318.9	150	168.9	1	0.804	0.888	0.493	0.261	83.2	252.1	402.1	0.804
Total	616.3	375	241.3	1	0.804	1.554	0.667	0.261	160.8	402.1	777.1	0.888
Prices of production: Previous Solution 3												
Depart. I	334.8	253.3	81.5	1.126	0.906	3.108	1.072	0.261	87.4	168.9	422.2	1.126
Depart. II	359.1	168.9	190.2	1.126	0.906	0.888	0.493	0.261	93.7	283.9	452.8	0.906
Total	693.9	422.2	271.7	1.126	0.906	1.554	0.667	0.261	181.1	452.8	875	1

$$\left(P_2 / D_2\right) = 1 \quad \text{or} \quad \left(m_c C_2 + m_v V_2\right)\left(1 + r\right) = C_2 + V_2\left(1 + e\right) \tag{31}$$

Given that equations (24), (25), and (26) are assumed by Bortkiewicz, it is necessary to provide a single equation to close the system. This "missing" equation is obtained by assuming *constant* real wages, an assumption that Bortkiewicz (1952: 20) attributed to Marx.[11] That is,

$$m_v = \left(P_2 / D_2\right) \quad \text{or} \quad m_v\left(C_2 + \left(1+e\right)V_2\right) = \left(m_c C_2 + m_v V_2\right)\left(1 + r\right) \tag{32}$$

After replacing equations (27) and (28), and therefore also equations (29) and (30), with equations (31) and (32), Bortkiewicz (1984: 208 *ff.*) provides several (rather unnecessary) examples in order to show that equations (28) and (29) are "false" simply because they (obviously) do not hold in his own scheme of simple reproduction. By starting from the SAV solution in Table 3, the main figures of the non-negative solution of the system of equations (24), (25), (26), (31), and (32) are also summarized in Table 3 (previous *solution 1*). Note that this solution makes the rate of profit an *endogenous* variable of the price-of-production system, because it depends on equilibrium equation (26), which destroys an interesting property of Marx's own solution; namely, that all industries contribute to the formation of the rate of profit according to the share of their capitals in aggregate capital. To the contrary, equation (32) allows for the (rather odd) possibility that certain industries (in modern usage "non-basic" sectors) do not contribute at all to the formation of the rate of profit.

Alternatively, *solution 1* can be reached by considering equations (24), (25), (26), (32), and the second invariance, or equation (30). This is largely the approach taken by Meek (1956), who suggests that aggregate surplus-value is unaffected by the transformation. However, Meek's numerical example does not satisfy equation (26), which forced him to assume the restriction $C_2/V_2 = C/V$ (*ibid.*: footnote 2, p. 103) in order to impose also constant nominal wages,

$$m_v = 1 \tag{33}$$

However, on restoring the equilibrium given by equation (26), it becomes clear that nominal wages remain constant whatever the organic composition of capital advanced in department II may be. This is the core argument of the solution proposed by Laibman (1973), which rests on the condition of "invariance of the share of wages in current labor time (*not* the share of wages in current value added); or, since by hypothesis current labor time is given and constant, invariance of the total wages in value terms (variable capital)" (*ibid.*: 224). Here, this proposition leads simply to the replacement of equation (31) of Bortkiewicz's system with the assumption of constant nominal wages, given by equation (33), which leads us to the same solution.

Finally, *solution 1* can also be generated by following the transformation method proposed by Foley (1982), Lipietz (1982), and Duménil (1983-84), which may be formalized by substituting equations (27) and (28) of the SPP simultaneous version with the assumption of constant nominal wages, given by equation (33), and with the following equation,

$$m_v V + \left(m_c C + m_v V\right) r = V\left(1 + e\right) \tag{34}$$

[11]As Baumol (1983: 303) points out: "I find few things as discouraging as the persistent attribution of positions to a writer whose works contain repeated, categorical, indeed emotional, denunciations of those views. Marx's views on wages are a prime example. Both vulgar Marxists and vulgar opponents of Marx have propounded two associated myths: that he believed wages under capitalism are inevitably driven near some physical subsistence level, and that he considered this to constitute robbery of the workers and a major evil of capitalism."

In this "new solution," aggregate value added (net output) remains unchanged in the transformation. Note that this solution does not necessarily rest on Bortkiewicz's assumption of constant real wages, since equation (34) taken by itself does not preclude *real* changes in income distribution. However, in the context of the two-sector model analyzed here, equations (33) and (34) taken together lead again to equations (31) and (32) and, for this reason, the "new solution" is exactly the same as the "old solution."

Solution 2. Since, according to Bortkiewicz (1952: 14), any of the available commodities can perfectly serve as measure of value, it is also feasible to ascribe the commodity that serves as money to department I instead of department II. In such a case, equation (31) in Bortkiewicz's system should be replaced by

$$\left(P_1 / D_1\right) = 1 \quad \text{or} \quad \left(m_c C_1 + m_v V_1\right)\left(1 + r\right) = C_1 + V_1\left(1 + e\right) \tag{35}$$

When the system formed by equations (24), (25), (26), (32), and (35) is solved by beginning from the SAV solution, it yields the outcomes reported in Table 3 (previous *solution 2*). The reduction in nominal wage rate imposed by this solution, $m_v = 0.804$, is accompanied by a reduction in price of department II's output of exactly the same size, $(P_2/D_2) = 0.804$, thereby rendering the workers' purchasing power unchanged. Hence, this mechanism of price assignation is nothing but a special version of the ancient "iron law of wages," a proposition strongly criticized by Marx.[12]

Solution 3. Note that "if we were to choose the price unit in such a way that total price and total value are equal" (Bortkiewicz 1984: 202), we should have to set nonetheless equation (29) instead of equation (31). The result is the system of equations (24), (25), (26), (29), and (32), whose non-negative solution, by starting again from the SAV solution, is summarized in Table 3 (previous *solution 3*). This solution was first analyzed by Winternitz (1948) and May (1948). Although all prices are now allowed to vary, the rate of profit also depends here on the equilibrium condition, or equation (26).

All previous solutions reported in Table 3, on the other hand, can easily be "iterated" by reintroducing the temporal dimension of involved variables into their corresponding system of equations. However, all these iterative solutions preclude Marx's laws of compensation from holding at the same time due to the prevailing assumption of constant real wages, or equation (32). For example, Anwar Shaikh provides a numerical illustration of the iterative method in the context of *solution 3* by assuming that "since the price of labor-power is determined by the price of its means of subsistence, the aggregate cost-price… is in effect the total price of means of production and means of subsistence" (Shaikh 1977: 125). This restrictive assumption also holds in the closely related work by Kliman and McGlone (1988), although the method of transformation proposed by these authors fails to yield a stationary equilibrium solution (see Laibman 2000: 323).

[12]"But this prejudice was first established as a dogma by the arch-philistine, Jeremy Bentham, that soberly pedantic and heavy-footed oracle of the 'common sense' of the nineteenth-century bourgeoisie…. This dogma… was used by Bentham himself, as well as by Malthus, James Mill, MacCulloch, etc., for apologetic purposes, and in particular so as to represent one part of capital, namely variable capital, or that part convertible into labour-power, as being of fixed size. Variable capital in its material existence, i.e. the mass of the means of subsistence it represents for the worker, or the so called labour fund, was turned by this fable into a separate part of social wealth, confined by natural chains and unable to cross the boundary to the other parts. To set in motion the part of the social wealth which is to function as social capital, or, to express it in a material form, as means of production, a definite mass of living labour is required. This mass is given by technology. But the number of workers required to put this mass of labour-power in a fluid state is not given, for it changes with the degree of exploitation of the individual labour-power. Nor is the price of this labour-power given, but only its minimum limit, which is moreover very elastic" (Marx 1976: 758-60).

Finally, it is important to stress that all previous solutions reported in Table 3 are *proportional* since they arise from the very same solution in terms of relative prices, as Samuelson (1970: 424-25, 1971: 425) so sharply demonstrates; in this case, "how one scales or normalizes... these numbers is an unessential issue even though it has given rise to some acrimonious and sterile debate among scholars." In contrast, the SPP solution cannot be generated by scaling Samuelson's formulae, precisely because such a solution does not rest on the assumption of constant (nominal and/or real) wage rates.

6. Some Concluding Remarks

In this paper, strong textual evidence, mostly reported in Table 1, has been provided which supports the view that the quantitative analysis of expanded reproduction, developed by Marx in chapter 21 of volume II of *Capital*, should be accurately represented by an identifiable system of finite-difference equations. On the basis of the inter-temporal equilibrium framework underlying this system, we have subsequently formalized the problem of transforming values into production prices by following the method outlined by Marx in chapter 9 of volume III of *Capital*. The numerical results derived from this formalization, reported in Table 2, suggest that Marx's own solution can be interpreted as the natural starting point of a chain of transformations along which successive transfers of surplus-value over time lead to the eventual formation of *full prices of production*. On this basis, the system of prices of production exhibits exactly the same path of capital accumulation and economic growth as the system of actual values, which leads us to conclude that, whatever the size of price-value deviations may be, values are indeed the *dominant determinants* of prices in the context of Marx's scheme of accumulation.

Throughout the discussion made in this paper the argument has been confined to conditions of competitive equilibrium, but an equilibrium only in the sense that the ceaseless deviations between supply and demand average out so as to be able to focus on more fundamental relations. As Marx (1991: 291) recalls, "the real inner laws of capitalist production clearly cannot be explained in terms of the interaction of supply and demand […] since these laws are realized in their pure form only when demand and supply cease to operate, *i.e.* when they coincide." In other words, the important thing is not equilibrium itself, but rather the role that equilibrium plays in the grounding of the laws of motion of the capitalist system. In this regard, the law of the equalization of profit rates analyzed in this paper does not depend whatsoever on prevailing equilibrium conditions, as we have seen, because the formation of a *general* rate of profit depends only on the same factors that determine the *average* rate of profit; to the contrary, equilibrium only can eventually take place in this system on the basis of a *given* rate of profit.

A basic result of the analysis developed in this paper is that a consistent solution of the transformation problem on the basis of Marx's laws of compensation requires the abandonment of the assumption of constant wage rates as a way to constrain the formation of commodity prices. As Marx claims in *Value, Wage and Profits* against similar arguments by one John Weston and his comrades: "reduced to their simplest theoretical expression, all our friend's arguments resolve themselves into this one dogma: *The prices of commodities are determined or regulated by wages*" (Marx 2000: 12). This is precisely the nature of Bortkiewicz's dogma, as we have shown by way of example in Table 3. Not surprisingly, once the (otherwise untenable) doctrine of "subsistence wages" is removed, and Marx's compensation laws are subsequently restored on the basis of volume II of *Capital*, then the legendary "great contradiction" between values of volume I and prices of volume III, first proclaimed by Eugen von Böhm-Bawerk (1984), simply vanishes.

Of course, it would be possible to consider Marx's scheme of reproduction from the point of view of the answer it provides to other relevant questions, such as fixed capital, technical change, distribution of income, public expenditure and taxes, credit-money and banking system, international exchange, unbalanced growth and unemployment, and economic cycles and crises.

However, each of these topics probably merits singular treatment, which is beyond the scope of this paper.

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